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Review of the Problems Solved Using the Constrained Bayesian Methods

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Abstract: The paper discusses a new philosophy of statistical hypothesis testing - the constrained Bayesian method (CBM) and its application to testing various types of hypotheses. In particular, to testing simple, composite, asymmetric, multiple hypotheses. The advantage of CBM over existing classical methods is theoretically proven in the form of theorems and practically demonstrated by the results of computing many examples. Examples of using CBM to solve some practically important problems are presented, which confirms the flexibility of the method and its great ability to cope with complex problems.

The article's findings help the educated public to familiarize themselves with the modern solutions of different important Problems Using the Constrained Bayesian Methods.

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MSC: 62F15, 62F03

1. Introduction

As is known, there are four main approaches (philosophies) to statistical hypothesis testing. The difference between them lies in the formalization of the problem, the metrics used, and the information they use. The authors of these approaches are: Fisher (Fisher, 1925), Neyman-Pearson (Neyman and Pearson, 1933), Jeffrey (Jeffreys, 1939) (so-called parallel methods), and Wald (so-called sequential method) (Wald, 1947a). Based on these ideas, many methods have been developed and are successfully used to solve various problems. The interested reader may find a fairly good and representative overview of modern methods for testing statistical hypotheses in the works (Kachiashvili, 2018; Kachiashvili, 2023), which we do not include here in order to avoid an unjustified overload of the volume of the work.

Since the late 1970s, we have been developing a new approach (philosophy) for testing hypotheses, which we have called the Constrained Bayesian Method (CBM) (Kachiashvili, 2018; Kachiashvili, 2023).

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It is known that two types of errors can be made when testing hypotheses. Rejecting the hypothesis when it is true and accepting it when it is not true. The probabilities that correspond to such decisions are called the Type I and Type II error rates, respectively.

In the classical Bayesian method, an unconstrained optimization problem is solved. In particular, a general loss function is minimized that includes both type I and type II errors. In contrast, in CBM, one type of error is minimized while imposing constraints on the other type of errors.

Depending on the constraints imposed, seven different tasks can be formulated. This allows us to flexibly select the appropriate task based on the specifics of the problem to be solved (Kachiashvili, 2018). To facilitate reference to the corresponding tasks, they are presented in (Kachiashvili, 2018) as Task 1, Task 2, ..., Task 7. The seven possible statements of the problem can be divided into two groups. In the first group, we combine the three tasks in which the type I error rates are limited and the type II error rates are minimized. In the second group, we combine the remaining four tasks in which the type II error rates are limited while minimizing the type I error rates. Below we will consider only two of them. In particular, task 2 and task 7, as representatives of these two groups.

The novelty of the proposed article lies in the fact that the review of constrained Bayesian methods in terms of its application and for the problems as in the present article is not known to us. However, these issues are very interesting and important from the point of view of both theoretical and practical application.

For clarity, let us present the formulations of the tasks named above for each of these two groups.

2. Constrained Bayesian Method

Let us introduce the following notations: $x \in R^n$ is n dimensional random variable with distribution density $p(x;\theta)$; $H_i : \theta_i \in \Theta_i$, $i = 1,2,...,S$, are S quantity testable hypotheses, where $\Theta_i \subset \Omega^m$, $i = 1,2,...,S$; Θ_i are non-intersecting subsets $\bigcup \Theta_i = \Omega^m$; Ω^m is m dimensional parameter space; $p(H_i)$ is a priori probability of H_i hypothesis; $p(x \mid H_i)$ is the density of the distribution of x under the fairness of H_i hypothesis; Γ_j is the area of acceptance of H_j hypothesis; $L_1(H_i, \delta_j(x) = 1)$ and $L_2(H_i, \delta_j(x) = 0)$ are losses caused by wrongly accepting and wrongly rejecting hypotheses. A detailed description of the conditional Bayes method can be found in (Kachiashvili, 2018; Kachiashvili, 2023). Below, we briefly present two possible formulations of this method.

2.1. Restrictions on the Conditional Probabilities of Accepting True Hypotheses (CBM 2)

The average loss caused by incorrectly accepting hypotheses is minimizing

$$r_\delta = \min_{\{\Gamma_j\}} \left\{ \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{\Gamma_j} L_1(H_i, \delta_j(x) = 1)\, p(x \mid H_i) dx \right\},$$

subject to

$$\sum_{j=1}^{S} \int_{R^n - \Gamma_j} L_2(H_i, \delta_j(x) = 0)\, p(x \mid H_i) dx \le r_2,\ i = 1,...,S. \qquad (1)$$

The solution of this problem gives

$$\Gamma_j = \left\{ x : \sum_{i=1}^{S} p(H_i) L_1(H_i, \delta_j(x) = 1) p(x \mid H_i) < \sum_{i=1}^{S} \lambda_i L_2(H_i, \delta_j(x) = 0) p(x \mid H_i) \right\},$$

where the Lagrange multipliers λ_i, $i = 1,\ldots,S$, are determined so that the equalities in (1) hold.

2.2. Restrictions on Posterior Probabilities of Rejecting True Hypotheses (CBM 7)

In this case, we are looking for the maximum of the average probability of accepting true hypotheses (let us call it the average power of the criterion)

$$G_\delta = \max_{\{\Gamma_j\}} \left\{ \sum_{i=1}^{S} p(H_i) \sum_{j=1}^{S} \int_{\Gamma_j} L_2(H_i, \delta_j(x) = 0) p(x \mid H_i) dx \right\},$$

at the following restrictions

$$\sum_{i=1}^{S} p(H_i) \int_{\Gamma_j} L_1(H_i, \delta_j(x) = 1) p(x \mid H_i) dx \le r_7, \ j = 1,\ldots,S . \qquad (2)$$

The Lagrange method gives us

$$\Gamma_j = \left\{ x : \sum_{i=1}^{S} p(H_i) L_2(H_i, \delta_j(x) = 0) p(x \mid H_i) > \lambda_j \sum_{i=1}^{S} p(H_i) L_1(H_i, \delta_j(x) = 1) p(x \mid H_i) \right\}$$

, $j = 1,\ldots,S$,

where, λ_j are determined so that the equalities in (2) hold.

3. General characterization of the CBM

CBM is a generalization of existing methodologies that, compared to existing methodologies, provides unique results in terms of optimality of the decisions made with a minimal number of observations. The constrained Bayesian methodology uses all the information that existing methodologies use and contains all the features of the formalization of existing methodologies, which is why it provides better results than existing methodologies.

Specifically, the constrained Bayesian methodology uses a loss function and a priori probabilities to make decisions, similar to the classical Bayesian methodology. Like the Neyman-Pearson methodology, it also uses a criterion significance level and is data-dependent, similar to the Fisher methodology. This has significantly increased the quality of decisions made by the constrained Bayesian methodology compared to other methodologies.

In CBM, unlike existing ones, decision-making regions have unique properties. In particular, in the decision-making space there are not only decision-making regions, but also regions of impossibility of making a decision. This provides a unique opportunity to solve the problems of parallel and sequential analysis with a single methodology, which significantly increases the theoretical and practical value of the methodology.

The constrained Bayes methodology was used to test various types of hypotheses, including simple, composite, asymmetric, and multiple hypotheses. The advantages of CBM over existing methods are proven in the form of theorems and demonstrated by the results of numerous example calculations (Kachiashvili, 2018; Kachiashvili, 2023). When testing the mentioned hypotheses, the constrained Bayesian methodology allows to obtain such decision rules, which, without any coercion, naturally allow to minimize practically all existing criteria of optimality of the decision, such as: type I, II and III error rates; false acceptance rate (FAR

); pure directional false discovery rate ($pdFDR$); mixed directional false discovery rate ($mdFDR$); Family-wise error rates of type I and type II ($FWERI$, $FWERII$) and others.

To illustrate what has been said, let us quote some results obtained using the mentioned methodology.

Let us introduce the concept of the summary risk (SR) for making an incorrect decision when testing hypotheses as the weighted sum of the probabilities of incorrect decisions, i.e.

$$r_S(\Gamma) = \sum_{i=1}^{S} \sum_{j=1, j\neq i}^{S} L(H_i, H_j)\, p(H_i) \int_{\Gamma_j} p(x \mid H_i) dx .$$

Here $L(H_i, H_j)$ is the sum function of losses caused by both types of errors.

It is evident that for given losses and probabilities, the decision regions determine the value of SR. Let us note: Γ^{CBM} and Γ^{B} are the regions of acceptance of hypotheses, respectively, in constrained Bayesian (CBM) and Bayesian (B) methods; $r_S(\Gamma^{CBM})$ *and* $r_S(\Gamma^{B})$ are, respectively, the summary risks (SR) in constrained Bayesian and Bayesian methods.

The following theorem is proved (Kachiashvili and Mueed, 2013; Kachiashvili, 2018).

Theorem 1. *For given losses and probabilities, SR of making incorrect decision in CBM is a convex function of* λ *with a maximum at* $\lambda = 1$. *By increasing or decreasing* λ *, SR decreases, and in the limit, i.e., at* $\lambda \to \infty$ *or* $\lambda \to 0$ *, SR tends to zero*.

From the theorem directly follows the following.

Corollary 1. *At the same conditions SR of making the incorrect decision in CBM is less or equal to SR of the Bayesian decision rule, i.e.,* $r_S(\Gamma^{CBM}) \le r_S(\Gamma^{B})$.

By analogy with to Corollary 1, it is easy to verify that the SR of incorrect decision making in CBM is not greater than the SR of the frequency method, that is, $r_S(\Gamma^{CBM}) \le r_S(\Gamma^{f})$ *, where* $r_S(\Gamma^{f})$ *is the SR of the frequentist method*.

The following theorem is proved (Kachiashvili and Mueed, 2013).

Theorem 2. *Let us assume that probability distributions* $p(x \mid H_i)$, $i = 1,\ldots,S$ *, are such that, at increasing number of repeated observations* n *, the entropy concerning distribution parameter* θ *, in relation to which the hypotheses are formulated, decrease. In such a case, for all given values* $0 < \alpha_1 < 1$ *and* $0 < \beta_1 < 1$ *there can always be found such a finite, positive integer number* n^* *that inequalities* $\alpha(\bar{x}_n) < \alpha_1$ *and* $\beta(\bar{x}_n) < \beta_1$*, where the number of repeated observations is* $n > n^*$*, take place.*

Here $\alpha(\bar{x}_n)$ and $\beta(\bar{x}_n)$ are the probabilities of errors of the first and the second types, respectively, in constrained Bayesian task when the decision is made on the basis of arithmetic mean of the observation results $\bar{x}_n$ calculated by n repeated observations. That is, the CBM allows us to make a decision by increasing information so that the levels of possible first and second type errors are restricted to any desired level α_1 and β_1 accordingly.

Let us present examples of the use of the CBM in cases of different types of hypotheses (simple, composite, asymmetric, multiple) and show its advantage over existing methods.

4. Simple Hypotheses

4.1. Parallel Methods

Example 1 (Kachiashvili, 2014). Suppose that $X_1, X_2, ..., X_n$ are independent identically distributed (i.i.d.) random variables with distribution law $N(x;\theta,1)$. Let us test the basic hypothesis $H_0 : \theta = -1$ against the alternative $H_A : \theta = 1$.

Below, when discussing examples, we assume that the hypotheses are a priori equally probable.

Table 1 presents the results of testing hypotheses using various criteria for various $\bar{x}$ calculated from $n = 4$ observations.

The results of the computation show that on the basis of $\bar{x} = 0$, i.e. $B(x) = 1$, the alternative hypothesis is accepted in the Berger (T^C and T^* criteria), Bayesian, and Neyman-Pearson methods when, in reality, there is no basis for such a decision (Kachiashvili, 2014). Both hypotheses are equally likely to be true or false.

In this respect, Fisher's p-value test is better because it rejects the hypothesis H_0 but says nothing about H_A. However, the p-value does not contain information about whether another, more correct hypothesis exists instead of H_0. For this case, CBM confirms that both hypotheses are equally true or equally false, that is it gives the most logical answer, because in such a case an unambiguous decision cannot be made. All the considered methods correctly make decisions with different error probabilities for other values of the statistic $\bar{x}$. But even in these cases, the advantage of CBM is obvious, because it gives us the error levels of Type I and Type II for which one of the testable hypotheses can be accepted according to the given statistic $\bar{x}$.

Table 1. Results of hypothesis testing for different values of $\bar{x}$.

$\bar{x}$	$B(x)$	For tests T^C, T^* and Bayes			For N-P test			For *p-value* test		For constrained Bayes test (CBM)			
		CEP $\alpha(B)$	CEP $\beta(B)$	AH*)	α	β	AH *)	*p-value*	AH*)	γ	$\alpha(\gamma,B)$	$\beta(\gamma,B)$	AH*)
0	1	0.5	0.5	H_A	0.02275	0.02275	H_A	0.02275	Reject H_0	$\gamma \le 0.0455$ $\gamma > 0.0455$	$\alpha \le 0.02275$ $\alpha > 0.02275$	$\beta \le 0.02275$ $\beta > 0.02275$	No one[1]) Both 2)
0.028	0.8	0.444(4)	0.555(5)	H_A	0.02275	0.02275	H_A	0.01989	Reject H_0	$\gamma \in [0.02,0.025]$ $\gamma > 0.025$ $\gamma < 0.02$	$\alpha \in [0.02,0.025]$ $\alpha < 0.02$ $\alpha > 0.025$	$\beta \in [0.02,0.025]$ $\beta > 0.025$ $\beta < 0.02$	H_A Both No one

0.25	0.14	0.1228	0.8772	H_A	0.02275	0.02275	H_A	0.00621	Reject H_0	$\gamma \in [0.007,0.05]$ $\gamma > 0.05$ $\gamma < 0.007$	$\alpha \in [0.009,0.06]$ $\alpha < 0.009$ $\alpha > 0.06$	$\beta \in [0.007,0.05]$ $\beta > 0.05$ $\beta < 0.007$	H_A Both No one
1	0.00034	0.00034	0.99966	H_A	0.02275	0.02275	H_A	0.00003	Reject H_0	$\gamma \in [0.0001,0.5)$ $\gamma \geq 0.5$	$\alpha \in (0.00003,0.38936]$ $\alpha \leq 0.00003$	$\beta \in [0.0001,0.5)$ $\beta \geq 0.5$	H_A Both
-0.25	7.389	0.88	0.1192	H_0	0.02275	0.02275	H_0	0.06681	Accept H_0	$\gamma \in [0.007,0.05]$ $\gamma > 0.05$ $\gamma < 0.007$	$\alpha \in [0.00920,0.06145]$ $\alpha < 0.00926$ $\alpha > 0.06145$	$\beta \in [0.007,0.05]$ $\beta > 0.05$ $\beta < 0.007$	H_0 Both No one
-1	2980.958	0.99966	0.00034	H_0	0.02275	0.02275	H_0	0.5	Accept H_0	$\gamma \in [0.0001,0.5)$ $\gamma \geq 0.5$	$\alpha \in (0.00003,0.38936]$ $\alpha \leq 0.00003$	$\beta \in [0.0001,0.5)$ $\beta \geq 0.5$	H_0 Both

*) Accepted hypothesis

1) neither hypothesis can be accepted.

2) Both hypotheses are equally likely to be true.

γ is the averaged probability of rejection of a true hypothesis.

$\alpha(\gamma, B)$ and $\beta(\gamma, B)$ are appropriately Type I and Type II error rates depending on γ and likelihood ratio $B(x)$.

4.2. Sequential Methods

Example 2 (Kachiashvili, 2014). Let us consider the scenario of Example 1, but assume that the data is obtained sequentially.

The results of the computation of 17 random samples generated by $N(x;1,1)$ sequentially processed are given in Table 3. The arithmetic mean of the observations $x_k, \ldots, x_m$ is designated by $\bar{x}_{k,m}$ there.

Table 2. Results of hypothesis testing for normal sampling.

n	Observation results x_i	$\bar{x}_{k,m}$	The Berger's test	The Wald's test	$\bar{x}_{k,m}$ for sequential test of Bayesian type	The sequential test of Bayesian type
1	1.201596	$\bar{x}_{1,3} = 0.6347$	H_A, $\alpha = 0.0217$, $\beta = 0.9783$.	H_A	$\bar{x}_1 = 1.2016$	H_A
2	0.043484				$\bar{x}_{2,3} = 0.3512$	H_A
3	0.658932					
4	0.039022	$\bar{x}_{4,6} = 0.8942$	H_A, $\alpha = 0.0047$, $\beta = 0.9953$.	H_A	$\bar{x}_{4,5} = 0.3280$	H_A
5	0.616960					
6	2.026540				$\bar{x}_6 = 2.02654$	H_A

7	-0.422764	$\bar{x}_{7,11}=0.2992$	H_A, $\alpha=0.0478$, $\beta=0.9522$.	H_A	$\bar{x}_{7,9}=0.1643$	H_A
8	0.562569				$\bar{X}_{10,11}=0.5015$	H_A
9	0.353047					
10	-0.123311					
11	1.126263				$\bar{x}_{12}=1.521061$	H_A
12	1.521061	$\bar{x}_{12}=1.521061$	H_A, $\alpha=0.0456$, $\beta=0.9544$.	H_A		
13	1.486411	$\bar{x}_{13}=1.4864$	H_A, $\alpha=0.0487$, $\beta=0.9513$.	H_A	$\bar{x}_{13}=1.486411$	H_A
14	-0.578935	$\bar{x}_{14,16}=0.5536$	H_A, $\alpha=0.0348$, $\beta=0.9652$.	H_A	$\bar{x}_{14,16}=0.55362$	H_A
15	0.623006					
16	1.616669					
17	1.754413	$\bar{x}_{17}=1.7544$	H_A, $\alpha=0.0291$, $\beta=0.9709$.	H_A	$\bar{x}_{17}=1.754413$	H_A
$\bar{n}$			2.43	2.43		1.7

The table shows that the Wald and Berger T^* tests give identical results, although the error probabilities in the Berger test are slightly lower than in the Wald test due to the fact that Berger calculated the error probabilities with a given value of the statistic. In the Wald criterion, they are determined by the decision thresholds. Out of 17 observations, correct decisions were made 7 times based on 3, 3, 5, 1, 1, 3, and 1 observations in both tests, and all decisions were correct. The average number of observations in making a decision is equal to 2.43. With the Bayesian-type sequential method (which is the same CBM in sequential experiment (Kachiashvili, 2016)), correct decisions for the same sample were made 10 times based on 1, 2, 2, 1, 3, 2, 1, 1, 3, and 1 observations, and all decisions were also correct. The average number of observations in making a decision is equal to 1.7.

5. Composite hypotheses

As in the cases of testing simple hypotheses discussed above, CBM is optimal for testing composite hypotheses in parallel and sequential experiments. It also easily overcomes the Lindley paradox that arises when testing a simple hypothesis against a complex hypothesis. These facts are shown theoretically and based on the computation of specific examples in the work (Kachiashvili, 2016).

Let us show the fact of overcoming the Lindley's paradox in CBM (Kachiashvili, 2016). Let us consider the normally distributed random variable ξ and by $D=\{x_1,x_2,...,x_n\}$ designate a sample of it. The null and alternative hypotheses have the following forms

$H_0:\xi\sim N(x\,|\,\mu,\sigma^2)$ and $H_A:\xi\sim N(x\,|\,\cdot,\sigma^2)$, $p(\mu\,|\,H_A)=N(\mu\,|\,\mu_1,\sigma_1^2)$, with a priori probabilities $p(H_0)=p$ and $p(H_A)=1-p$.

The sample mean $\bar{x}$ is sufficient statistics and the posterior probability of the null hypothesis (on the basis of which a decision is made in Bayes method) is

$$p(H_0\,|\,D)=p(H_0\,|\,\bar{x})=\left(1+\frac{1-p}{p}\frac{p(\bar{x}\,|\,H_A)}{p(\bar{x}\,|\,H_0)}\right)^{-1}, \qquad (3)$$

where

$$p(\bar{x}\,|\,H_0)=N\left(\bar{x}\,|\,\mu_0,\sigma^2/n\right),$$

$$p(\bar{x}\,|\,H_A)=\int N\left(\bar{x}\,|\,\mu,\sigma^2/n\right)N\left(\mu\,|\,\mu_1,\sigma_1^2\right)d\mu=N\left(\bar{x}\,|\,\mu_1,\sigma_1^2+\sigma^2/n\right).$$

The final form of (3) is

$$p(H_0\,|\,D)=\left[1+\frac{1-p}{p}\left(\frac{\sigma^2/n}{(\sigma_1^2+\sigma^2/n)}\right)^{1/2}\frac{\exp\left\{(\bar{x}-\mu_0)^2/\left(2\sigma^2/n\right)\right\}}{\exp\left\{(\bar{x}-\mu_1)^2/\left(2\sigma_1^2+2\sigma^2/n\right)\right\}}\right]^{-1}. \qquad (4)$$

It is not difficult to see from (4) that, when $\bar{x}$ and p are fixed and σ_1^2 increases the right hand side of (4) tends to one, i.e. independently of the data values and a priori probability p, the null hypothesis will always be accepted for sufficiently large variance σ_1^2. This fact is called Lindley's statistical paradox.

Let us apply one of possible statements of CBM when the averaged probability of acceptance of true hypotheses is restricted for testing hypotheses (Kachiashvili, 2018). When the number of hypotheses is equal to two, this problem has the following form

$$\min_{\{\Gamma_0,\Gamma_A\}}\left\{p(H_0)\int_{\Gamma_A}p(\bar{x}\,|\,H_0)d\bar{x}+p(H_A)\int_{\Gamma_0}p(\bar{x}\,|\,H_A)d\bar{x}\right\},$$

at

$$p(H_0)\int_{\Gamma_0}p(\bar{x}\,|\,H_0)d\bar{x}+p(H_A)\int_{\Gamma_A}p(\bar{x}\,|\,H_A)d\bar{x}\geq 1-\alpha\,. \qquad (5)$$

Hypotheses H_0 and H_A acceptance regions respectively Γ_0 and Γ_A have the following forms in this case

$$\Gamma_0=\left\{\bar{x}:\frac{p(\bar{x}\,|\,H_A)}{p(\bar{x}\,|\,H_0)}<\lambda\frac{p}{1-p}\right\},\ \Gamma_A=\left\{\bar{x}:\frac{p(\bar{x}\,|\,H_0)}{p(\bar{x}\,|\,H_A)}<\lambda\frac{1-p}{p}\right\}.$$

For the considered example, we have

$$\Gamma_0 = \left\{ \bar{x} : \frac{1-p}{p} \left(\frac{\sigma^2/n}{\sigma_1^2 + \sigma^2/n} \right)^{1/2} \frac{\exp\left\{ (\bar{x}-\mu_0)^2 / (2\sigma^2/n) \right\}}{\exp\left\{ (\bar{x}-\mu_1)^2 / (2\sigma_1^2 + 2\sigma^2/n) \right\}} < \lambda \right\}, \qquad (6)$$

$$\Gamma_A = \left\{ \bar{x} : \frac{1-p}{p} \left(\frac{\sigma^2/n}{\sigma_1^2 + \sigma^2/n} \right)^{1/2} \frac{\exp\left\{ (\bar{x}-\mu_0)^2 / (2\sigma^2/n) \right\}}{\exp\left\{ (\bar{x}-\mu_1)^2 / (2\sigma_1^2 + 2\sigma^2/n) \right\}} > 1/\lambda \right\}. \qquad (7)$$

It is clear that the left-hand-side member of inequalities of Γ_0 and Γ_A tend to zero when σ_1^2 increases for any fixed $\bar{x}$ and p. In the classical Bayesian statement when $\lambda = 1$, inequality in (6) will always be satisfied and that in (7) will never be satisfied for a large variance σ_1^2. That means hypothesis H_0 will always be accepted.

In CBM, Lagrange multiplier λ in decision making regions (6) and (7) are chosen so that in restrictions (5) equalities take place, i.e. the average power of the test will always be equal to the level $1-\alpha$. That means that the value of λ depends on the value of α and it compensates for the influence of σ_1^2 on the left sides of the expressions in (6) and (7). The value of λ decreases when σ_1^2 increases for satisfying condition (5). Therefore, the Lindley's paradox will never arise because for the preservation of condition (5) the sizes of hypotheses accepting regions Γ_0 and Γ_A will be kept the same.

The value of λ is defined from condition (5) for the given values of μ_0, μ_1, σ^2, σ_1^2 and α, i.e. to the concrete value of λ correspond the concrete values of μ_0, μ_1, σ^2, σ_1^2 and α. From here it is seen that, when $\bar{x} \to \mu_0$, the probability of fulfilling of the inequality in (6) increases and the same probability in (7) decreases. Thus, in such a case, the probability acceptance of hypothesis H_0 increases, and the probability of acceptance of hypothesis H_A decreases independently of the value of σ_1^2. On the contrary, when $\bar{x} \to \mu_1$, the opposite situation exists: the probability the inequality in (7) increases and that in (6) decreases. This is the same as the probability of accepting hypothesis H_A increases, while the probability of accepting H_0 decreases.

6. Testing Multiple Hypotheses

Over the past few decades, the problem of multiple hypothesis testing has received considerable attention due to its widespread application (see, for instance, (O'Brien, 1984; Pocock et al., 1987; Shaffer, 1995; Dudoit et al., 2003; Tartakovsky and Veeravalli, 2004; De and Baron, 2012a,b and so on).

CBM maintains optimal properties when testing multiple hypotheses. For example, d individual hypotheses about parameters $\theta_1,\ldots,\theta_d$ of sequentially obtained vectors $\mathbf{X_1}, \mathbf{X}_2,\ldots$ were considered in (Kachiashvili, 2015)

$$H_0^{(1)} : \theta_1 \in \Theta_{01} \quad vs \quad H_A^{(1)} : \theta_1 \in \Theta_{11},$$

$$H_0^{(2)} : \theta_2 \in \Theta_{02} \quad vs \quad H_A^{(2)} : \theta_2 \in \Theta_{12},$$

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$H_0^{(d)}: \theta_k \in \Theta_{0d}$ *vs* $H_A^{(d)}: \theta_k \in \Theta_{1d}$.

The following stopping rule in the sequential test was developed

$$T = \inf\left\{ m : \bigcap\nolimits_{j=1}^{d} \left\{ \Lambda_m^{(j)} \notin \left(\lambda_{*j}, \lambda_j^* \right) \right\} \right\},$$

where λ_{*j} and λ_j^* are Lagrange multipliers obtained in CBM; $\Lambda_m^{(j)}$ is the likelihood ratio (8) for the j th parameter ($j = 1,\ldots,d$) computed on the basis of m sequentially observed vectors

$$\Lambda_m = \frac{p(\mathbf{x}^1 \mid H_A) \cdots p(\mathbf{x}^m \mid H_A)}{p(\mathbf{x}^1 \mid H_0) \cdots p(\mathbf{x}^m \mid H_0)},\ m = 1,2,\ldots \qquad (8)$$

The following theorem was proved (Kachiashvili, 2015)

Theorem 3. A *sequential Bonferroni procedure for testing multiple hypotheses (3) with stopping rule (4), rejection regions* $\Lambda_T^{(j)} > \lambda_j^*$, *and acceptance regions* $\Lambda_T^{(j)} < \lambda_{*j}$, *where* λ_j^* *and* λ_{*j} *are defined in CBM for* $\beta_j = \beta / d$ *and* $\alpha_j = \alpha / d$, *controls both error rates at levels* $FWER_I \le \alpha$ *and* $FWER_{II} \le \beta$.

The computed results (given in (Kachiashvili, 2015)) distinctly demonstrate the fairness and optimality of the developed scheme compared to existing methods.

7. Testing Multiple Hypotheses with Directional Alternatives in Sequential Experiments

The problem of testing a simple basic hypothesis versus composite alternatives is a frequently considered and well-studied problem in many scientific works (Wijsman, 1967; Anderson, 1982; Berger and Pericchi, 1996; Kass and Wasserman, 1996; Marden, 2000; Gomez-Villegas et al., 2009; Bedbur et al., 2013; Duchesne and Francq, 2014). In many situations, accepting alternative hypotheses raises the necessity to decide if the parameter's value outstrips or falls behind the value defined by the basic hypothesis (Kaiser, 1960; Leventhal and Huynh, 1996; Finner, 1999; Jones and Tukey, 2000; Shaffer, 2002; Bansal and Sheng, 2010; Bansal et al., 2016). Such problems arise, for example, in microarray data analysis, imaging analysis, biological applications and genetic research (Bansal and Sheng, 2010; Bansal and Miescke, 2013) and many others.

This problem can be stated as follows for parametric models

$$H_0: \theta = \theta_0 \text{ vs. } H_-: \theta < \theta_0 \text{ or } H_+: \theta > \theta_0, \qquad (9)$$

where θ is the parameter of the model, θ_0 is known. These alternatives are called skewed or directional alternatives.

Multiple directional hypotheses are considered in many applications, i.e. the following hypotheses are tested (Finner, 1999; Shaffer, 2002; Bansal et al., 2016; Bansal and Miescke, 2013; Benjamini and Hochberg, 1995)

$$H_i^{(0)}:\theta_i=0 \quad vs \quad H_i^{(-)}:\theta_i<0 \text{ or } H_i^{(+)}:\theta_i>0,\ i=1,...,m. \qquad (10)$$

where m is the number of individual hypotheses concerning parameters $\theta_1,...,\theta_m$. They must be tested using test statistics $\mathbf{X}=(X_1,X_2,...,X_m)$, where $X_i \approx f(x_i \mid \theta_i)$.

The following theorem for hypotheses (9) is proved in (Kachiashvili et al., 2020).

Theorem 4. *Let us assume that the probability distributions* $p(x \mid H_i)$, $i \in \Psi$, *where* $\Psi \equiv \{-,0,+\}$, *are such that, at increasing number of observations* n *in the sample, the entropy concerning distribution parameter* θ, *in relation to which the hypotheses are formulated, decreases. In such case, for given set of hypotheses* H_0, H_- *and* H_+, *there always exists such sample of size* n *on the basis of which decision concerning tested hypotheses can be made with given reliability when Lagrange multipliers are determined for* $n=1$ *in decision making regions and the condition* $mdFDR \le q$ *is satisfied.*

For multiple directional hypotheses (10) the stopping rule is as follows:

$$T=\inf\left\{n:\bigcap_{i=1}^{m}\left[t_i\left(\mathbf{X^{(1)}},\mathbf{X^{(2)}},...,\mathbf{X^{(n)}}\right)\in only\ \ one\ \ of\ \ \Gamma_j^i\right], j\in\mathbf{\Psi}\right\},$$

where $\mathbf{X}^{(i)}$, $i=1,...,n$, are independent m-dimensional observation vectors with correlated parameters; Γ_j^i is j-th hypothesis acceptance region for i-th sub-set of hypothesis.

The following theorem is proved in (Kachiashvili et al., 2020).

Theorem 5. *Let us for each sub-set of directional hypotheses of multiple hypotheses (10) one of the possible stated task of CBM is used with the restriction levels providing the condition* $mdFDR_i \le q_i$, $i=1,...,m$. *Then, if* q_i, $i=1,...,m$, *are taken so that* $\sum_{i=1}^{m} q_i = q$, *the following condition* $tmdFDR \le q$ *is fulfilled*.

Here $tmdFDR$ is the total mixed directional false discovery rate that was introduced in (Kachiashvili et al., 2020).

8. Testing Composite Hypotheses Concerning Normal Distribution with Equal Parameters

A normal distribution with equal parameters, that is $N(\theta,\theta)$, may be applied as a useful model for data In many practical situations (Mukhopadhyay and Zhang, 2018, 2020; Kachiashvili et al., 2024).

Let us consider a random variable X distributed as $N(\theta,\theta)$. The problem is to test the hypotheses

$H_0:\theta=\theta_0$, $\theta_0>0$ vs. $H_-:0<\theta<\theta_0$ or $H_+:\theta>\theta_0$,

using *independent identically distributed* (i.i.d.) random variables $X_1, X_2,..., X_n$.

The following theorems for directional alternatives were proved in (Kachiashvili et al., 2020).

Theorem 6. *CBM 7, at satisfying a condition* $\frac{r_7^- + r_7^+}{K_1 \cdot P_{\min}} = q$ *(i.e.* $r_7^- + r_7^+ = q \cdot K_1 \cdot P_{\min}$*), where* $0<q<1$, $P_{\min} = \min\{p(H_-), p(H_0), p(H_+)\}$*, ensures a decision rule with mdFDR (i.e. with* $SERR_{III}$*) less or equal to* q*, i.e. with the condition* $mdFDR = SERR_{III} \le q$.

Theorem 7. *CBM 7, at satisfying a condition* $\frac{r_7^- + r_7^0 + r_7^+}{K_1 \cdot P_{\min}} = q$ *(i.e.* $r_7^- + r_7^0 + r_7^+ = q \cdot K_1 \cdot P_{\min}$*), where* $0<q<1$, $P_{\min} = \min\{p(H_-), p(H_0), p(H_+)\}$*, ensures a decision rule with* $mdFDR + FAR$ *less or equal to* q*, i.e. with the condition* $mdFDR + FAR \le q$.

Here, we have: $FAR = P(x \in \Gamma_0 \mid H_-) + P(x \in \Gamma_0 \mid H_+)$ as the false acceptance rate; $SERR_{III}$ is the summary type III error rate; r_7^-, r_7^0, r_7^+ are restriction levels in CBM 7; K_1 is the loss for incorrect acceptance of hypothesis.

Moreover, the following theorems are proven in (Kachiashvili et al., 2024).

Theorem 8. *CBM 2, for hypotheses (10), ensures a decision rule with the error rates of the Type-I and Type-II restricted by the following inequalities*

$$\alpha \le \frac{r_2^0}{K_0 \cdot p(H_0)},$$

$$\beta \le \frac{r_2^-}{K_0 \cdot p(H_-)} + \frac{r_2^+}{K_0 \cdot p(H_+)}.$$

Theorem 9.*CBM 2, for hypotheses (10), ensures a decision rule with* r_δ *the averaged loss of incorrectly accepted hypotheses and* α_s *the averaged loss of incorrectly rejected hypotheses restricted by the following inequalities*

$$r_\delta \le \frac{K_1}{K_0}\left[r_2^- + r_2^0 + r_2^+\right],$$

$$\alpha_s \le r_2^- + r_2^0 + r_2^+.$$

Correctness of these theorems and superiority of CBM over Bayes method are clearly demonstrated by computation results.

9. Testing Equi-Correlation Coefficient

Symmetric Multivariate Normal Distribution (SMND) is widely used in many applications of different spheres of human activities such as psychology, education, genetics and so on (SenGupta, 1987). It is also used extensively in statistical inference procedures, e.g., in analysis of variance (ANOVA) for modeling of the error part (De and Mukhopadhyay, 2019).

Let consider k-dimensional random vector X which has normal distribution with zero mean vector and correlation matrix W with the following structure:

$$W = (1-\rho)\cdot I_{k\times k} + \rho \cdot J_{k\times k},$$

where $I_{k\times k}$ is an identity matrix and $J_{k\times k}$ is matrix of ones.

The support of ρ, $\left(-1/(k-1),1\right)$ assures the non-singularity of the correlation matrix W (De and Mukhopadhyay, 2019).

The problem can be formulated as follows: to test

$$H_0 : \rho = \rho_0 \text{ vs. } H_- : -1/(k-1) < \rho < \rho_0 \text{ or } H_+ : \rho_0 < \rho < 1, \qquad (11)$$

$$\rho, \rho_0 \in \left(-1/(k-1),1\right),$$

on the basis of the sample $Y_1,...,Y_n$. Here Y_i is the Helmert orthogonal transformation $Y_1, Y_2,...$ on uncorrelated multivariate normal vectors sequence $X_1, X_2,...$, i.e. $Y_i = H \cdot X_i$, $i = 1,2,...,$ where H is the Helmert matrix given as (De and Mukhopadhyay, 2019)

$$H = \begin{bmatrix} \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & & ... & \frac{1}{\sqrt{k}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & & ... & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 & ... & 0 \\ ..... & ..... & ..... & ..... & ..... & ..... \\ \frac{1}{\sqrt{k(k-1)}} & ... & \frac{1}{\sqrt{k(k-1)}} & & -\frac{(k-1)}{\sqrt{k(k-1)}} & \end{bmatrix}.$$

The k-dimensional random vector Y_i has independent components. The components have zero means and variances determined by

$v_1 = 1+(k-1)\rho$ and $v_2 = ... = v_k = 1-\rho$.

The probability density function (pdf) of Y_i is

$$p(Y_i \mid \rho) = (2\pi)^{-\frac{k}{2}} |V|^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2} Y_i \cdot V^{-1} \cdot Y_i^T\right\},\ i = 1,...,n.$$

The joint pdf of the sample $Y_1,...,Y_n$ has the following form (SenGupta and Gokhale, 1986)

$$p(Y_1,...,Y_n \mid \rho) = (2\pi)^{-kn/2} \cdot |V|^{-\frac{n}{2}} \cdot \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} Y_i \cdot V^{-1} \cdot Y_i^T\right\} =$$

$$= (2\pi)^{-kn/2}(1+(k-1)\rho)^{-\frac{n}{2}}(1-\rho)^{-\frac{n(k-1)}{2}} \cdot \exp\left\{-\frac{1}{2}\left(\frac{V_{1n}}{1+(k-1)\rho} + \frac{V_{2n}}{1-\rho}\right)\right\}, \qquad (12)$$

where V_{1n} and V_{2n} are defined by ratio

$V_{1n} = \sum_{i=1}^{n} y_{i1}^2$ and $V_{2n} = \sum_{i=1}^{n}\sum_{j=2}^{k} y_{ij}^2$,

where $Y_i = (y_{i1}, y_{i2},..., y_{ik})$, $i = 1,...,n$.

To test hypotheses (11) were applied CBM using the maximum likelihood estimation and Stein's approach (Kachiashvili and SenGupta, 2024). Using Stein's approach simulation results showed that CBM allows to make decisions with higher reliability for testing (11) hypotheses then Bayes method (see Figure 1).

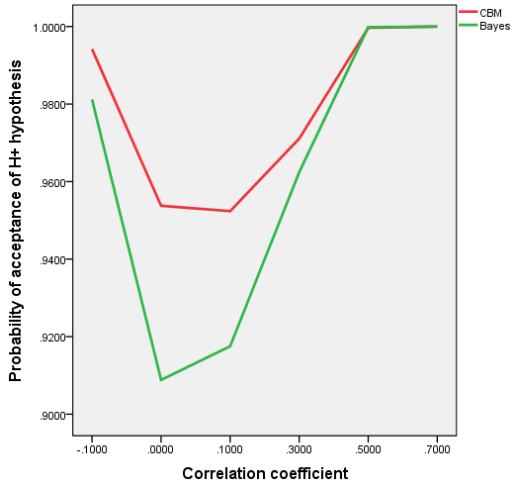

Fig. 1. Dependence of true Hypothesis H_+ acceptance probability on correlation coefficient.

The samples for making decisions are generated by (12) with $\rho \in (\rho_0, 1)$ for the size $m = 5000$.

10. Solving ANOVA problem with restricted Type I and Type II error rates

The comparison of the means of several normally distributed random variables deserved researchers' attention since the middle of last century. Similar problems arise in many practical applications. For instance, at comparing treatment effects in clinical trials, or experimental studies at comparing effects of different experiments, or at comparing the failure rate of a machine with its ages (Kachiashvili et al., 2025). Also, an important question in many applications is whether the nonparametric time trends are all the same (Khismatullina, 2022). In the framework of ANOVA, the formulation of the problem can be presented as follows (Aghamohammadi et al., 2010). Let us consider k normal populations $N(\mu_i, \sigma_i^2)$, $i = 1,...,k$. Let $X_{i1},..., X_{in_i}$ be independent from each other random sample from $N(\mu_i, \sigma_i^2)$. The problem is to test hypotheses

$$H_0 : \mu_1 = \ldots = \mu_k \text{ vs. } H_A : \mu_i \neq \mu_j \text{ for some } i \neq j . \qquad (13)$$

In some investigations there are considered cases when before observing the data it is supposed that the true means are simply ordered ($\mu_1 \leq \ldots \leq \mu_k$). For instance, the treatment may be organized as a sequence of increasing dose levels of a drug. In this case ANOVA takes the form

$$H_0 : \mu_1 = \ldots = \mu_k \text{ vs. } H_A : \mu_1 \leq \ldots \leq \mu_k , \qquad (14)$$

with at least one strict inequality. This case has been discussed in much of the statistical literature (see, for example, (Aghamohammadi et al., 2010; Barlow et al., 1972; Hayter, 1990; Ichiba, 2016; Robertson et al., *1988*)).

The case $k = 2$ when two normal populations are considered is called the Behrens-Fisher problem and it is well studied. A simple and accurate method is introduced in (Welch, 1947, 1951) for its solution, for the statement (13). It gives good results in both cases when variances are equal or are not. A quite good review of the existing methods of solving the problem (13) is given in (Krishnamoorthy et al., 2007) [52] for arbitrary cases, when $k > 2$. There is concluded that a procedure that gives satisfactory results by Type I error rate for all sample sizes, large and parameters configurations does not exist (Krishnamoorthy et al., 2007; Dajani, 2002).

A one-sided test is offered in Hayter (Hayter, 1990) for the statement (14). It provides simultaneous one-sided lower confidence bounds for the ordered pairwise comparisons $\mu_i - \mu_j , 1 \leq j < i \leq k$. Similar inference procedures are discussed in many works.

Without loss of generality, we can suppose that all μ_i ($i = 1,\ldots,k$) are positive, because if this condition is not fulfilled, we can always achieve it by moving the origin.

Let us rewrite hypotheses (13) as follows

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_{k-1} = 0 \text{ vs. } H_A : \theta_i \neq 0 , \text{ for some } i \in (1,\ldots,k-1) , \qquad (15)$$

where $\theta_i = \mu_{i+1} - \mu_i$, $i = 1,\ldots,k-1$. In this case, hypotheses (14) take the form

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_{k-1} = 0 \text{ vs. } H_A : \theta_i \geq 0 , \text{ with at least one strict inequality.} \qquad (16)$$

Let us to use the concept of directional hypotheses (Kachiashvili, 2018) and consider hypotheses (15) as a set of directional hypotheses

$$H_0^i : \theta_i = 0 \text{ vs. } H_-^i : \theta_i < 0 \text{ or } H_+^i : \theta_i > 0 , \; i = 1,\ldots,k-1 . \qquad (17)$$

Statement (17) allows us to test hypotheses (13) and (14) (which are the same as hypotheses (15) and (16)) simultaneously, using techniques developed in (Kachiashvili, 2023; Mukhopadhyay and Zhang, 2018). In practice, after testing hypotheses (17), if for even one value of i , left sided hypothesis is accepted among accepted alternatives, accept alternative hypothesis in (15), if all the accepted alternative hypotheses are right sided – accept alternative hypothesis in (16), otherwise accept the null hypothesis.

To test a subset of individual hypotheses of (17) the following theorem was proven.

Theorem 10. *CBM 7, at satisfying a condition* $(r_7^- + r_7^+)/(K_1 \cdot p(H_0)) = q$, *where* $0 < q < 1$ *, ensures a decision rule with Type I error rate less or equal to* q *, i.e. with the condition* $TIER \leq q$.

For testing multiple directional hypotheses (*17*), the concept of the $tmdFDR$ was used

$tmdFDR = \sum_{i=1}^{k-1} mdFDR_i$.

To provide a level q for the total Type I error rate ($TTIER$) and for the total Type II error rate ($TTIIER$), we provide q_i , the level of the appropriate criteria for the i th subset of the individual directional hypotheses. As a result, we have

$$TTIER = \sum_{i=1}^{k-1} TIER_i \text{ and } TTIIER = \sum_{i=1}^{k-1} TIIER_i ,$$

where $TIER_i$ and $TIIER_i$ are the Type I error rate and the Type II error rate, respectively, of the i th subset of directional hypotheses (Kachiashvili, 2015; Kachiashvili et al., 2020).

The values of q_i in all three cases (for $tmdFDR$, for $TTIER$ and for $TTIIER$) can be chosen to be equal, i.e. $q_i = q/(k-1)$ or different, e.g. inversely proportional to the informational distances between the tested hypotheses in the subsets of directional hypotheses (Kachiashvili et al., 2020).

The proposed CBM-based method for solving the ANOVA problem, for known and unknown variations of the observed results, with restricted Type I and Type II error rates, provides very reliable results that are superior to currently available methods. The latter is confirmed by the results of computations obtained on real and simulated data in various scenarios (Kachiashvili et al., 2025).

The offered method is a sequential one that requires a set of observations to make a decision.

11. Conclusion

A brief review of the application of CBM to test different types of statistical hypotheses arising in many practical problems is presented above. It confirms the advantage of the CBM over existing classical methods at testing simple, composite, asymmetric, multiple hypotheses of different kinds. Obtained results, published in many important scientific publication, confirm the flexibility of the method and its great ability to deal with difficult problems.

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