

UDC 545-1

Nonlocal boundary value problem for a fractional partial differential equation in a domain with a curved boundary

A. V. Pskhu

Abstract: A nonlocal boundary value problem in a domain with a curved boundary is studied for a first-order partial differential equation with a fractional Riemann-Liouville derivative with respect to one of the two independent variables. The nonlocal condition is specified as an integral of the desired solution. A theorem on the existence, uniqueness, and representation of the solution to the problem is proved.

DOI: 10.5281/zenodo.20684763

Keywords: First-order partial differential equation; Riemann-Liouville derivative.

1. Introduction

Throughout this note, all rings are assumed to be associative with identity unless stated otherwise. For a ring R , the polynomial ring in one indeterminate x over R is denoted by $R[x]$. The full $n \times n$ matrix ring over R is denoted by $M_n(R)$, and the $n \times n$ upper triangular matrix ring over R is denoted by $T_n(R)$. The matrix in $M_n(R)$ whose (i, j) -entry is 1 and all other entries are 0 is denoted by e_{ij} . We denote by $S_n(R)$ the subring of $T_n(R)$ in which all diagonal entries are equal, and by $V_n(R)$ the subring of $S_n(R)$ consisting of those matrices a that satisfy $a_{ij} = a_{i+k, j+k}$ for all $1 \leq i \leq n$ and $1 \leq k \leq n - j$.

For a given ring R , the Jacobson radical, the prime radical, and the set of all nilpotent elements of R are denoted by $J(R)$, $P(R)$, and $N(R)$, respectively. Recall that a ring R is called semiprime, semiprimitive, reduced, and 2-primal if, respectively, $P(R) = 0$, $J(R) = 0$, $N(R) = 0$, and $P(R) = N(R)$ (see [1–3]).

Recall from [4] that for a commutative ring R , an R -module M , and an endomorphism σ of R , the Nagata extension of R by M and σ is the ring $R \oplus M$ with component-wise addition and multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1) m_2 + r_2 m_1)$ for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. If σ is the identity mapping, the extension is called the trivial extension of R by M , denoted by $T(R, M)$. In particular, $T(R, R) \cong S_2(R)$.

In [5], a ring R is called reflexive, if $aRb = 0$ if and only if $bRa = 0$ for all $a, b \in R$.

Cohn [6] introduced a stronger version of reflexivity and called a ring R reversible if $ab = 0$ if and only if $ba = 0$. Reversible rings are also known as zero-commutative by Habeb [7], and as rings in which zero products commute by Anderson [8]. A ring R is said to have the insertion-of-factors property (IFP) if for every $a, b \in R$, the condition $ab = 0$ implies $aRb = 0$.

The concept of IFP has appeared under several different names: De Narbonne [9] called such rings semicommutative, Shin [10] used the term (SI) condition, and Habeb [7] referred to them as zero insertive.

In [11], Kwak et al. proved that the trivial extension $R = T(D, D)$ of any division ring D satisfies the condition $aRb = bRa$ for all $a, b \in R$. However, this condition does not hold, in general; for example, it may fail when D is a commutative ring that is not a division ring.

This observation motivated them to introduce the concept of axis-commutativity of rings. A ring R is said to be axis-commutative if $aRb = bRa$ for all $a, b \in R$.

Although axis-commutativity does not always hold, we have the following example showing that the condition is satisfied under certain additional assumptions.

Example 1. Let $S = F\langle x, y \rangle$ be the free algebra generated by the noncommuting indeterminates x and y over a field F . Then S is a domain which is not a division ring. Consider the ring $R = T(S, S)$. It is shown in ([11] [Example 1.2]) that R is not axis-commutative. However, if $a = (\alpha_1, \alpha_2)$ and $b = (\beta_1, \beta_2)$ are elements of R with $ab = 0$, then $\alpha_1\beta_1 = 0$, which implies that either $\alpha_1 = 0$ or $\beta_1 = 0$. In both cases, we obtain $aRb = bRa$.

The preceding discussion motivates us to introduce a generalization of axis-commutative rings. We call a ring R axis-reversible if $aRb = bRa$ for all $a, b \in R$ whenever $ab = 0$.

The main contributions of this paper are as follows: We introduce and systematically

study the class of axis-reversible rings as a natural generalization of axis-commutative rings. We investigate their relationships with several well-known classes of rings, including reversible, semicommutative, reduced, semiprime, and 2-primal rings. In particular, we establish nontrivial implications and characterizations involving axis-reversibility and show that this property is independent from semicommutativity through explicit counterexamples. We further study the behavior of axis-reversible rings under various constructions such as trivial extensions, direct products, corners, and Nagata extensions. Finally, we classify minimal noncommutative axis-reversible rings and describe their structure, providing new insight into the role of axis-

reversibility in noncommutative ring theory. These results provide a framework for further investigations of axis-related properties in noncommutative ring theory.

Definitions and Examples

We begin this section with the following definition.

Definition 1. A ring R (not necessarily with identity) is called axis-reversible if for all $a, b \in R$, $ab = 0$ implies $aRb = bRa$.

Clearly, every axis-commutative ring is axis-reversible. However, the converse is not necessarily true, as shown in Example 1. The next theorem gives some examples of axis-reversible rings.

Theorem 1. For any ring R , the following assertions:

- (i) If $T(R, R)$ is axis-reversible, then R is so.
- (ii) If R is reduced, then $T(R, R)$ is axis-reversible.
- (iii) $S_n(R)$ is not axis-reversible, for every integer $n \geq 3$;
- (iv) $M_n(R)$ and $T_n(R)$ are both not axis-reversible, for every integer $n \geq 2$.
- (v) If $V_n(R)$ is axis-reversible, for some $n > 1$, then R is so.
- (vi) If R is reduced, then $V_n(R)$ is axis-reversible for every $n > 1$.

Proof. (i) Assume that $T(R, R)$ is axis-reversible, for some ring R . Let $ab = 0$ for some $a, b \in R$, and define the elements $\alpha = (a, 0)$ and $\beta = (b, 0)$ in $T(R, R)$. Then $\alpha\beta = 0$. Hence, $\alpha T(R, R)\beta = \beta T(R, R)\alpha$, which implies that $aRb = bRa$. Therefore, R is axis-reversible.

(ii) Let $\alpha = (a, b)$ and $\beta = (c, d)$ be nonzero elements of $T(R, R)$ such that $\alpha\beta = 0$. Then, $ac = 0$ and $ad + bc = 0$. Since R is reversible by ([12] [Theorem 2.5]), we have $aRc = cRa = 0$. Moreover, $(ad)^2 = (-bc)^2 = bc bc = -bcad = 0$, and the reducedness of R implies $ad = 0$. Similarly, $bc = 0$. Therefore, $\alpha T(R, R)\beta = \beta T(R, R)\alpha = 0$, and hence $T(R, R)$ is axis-reversible.

(iii) Consider the elements $a = e_{23}$ and $b = e_{12}$ in $S_n(R)$. We have $ab = 0$ while

$aS_n(R)b = 0$ and $bS_n(R)a = S_n(R)e_{13}$. Therefore, $S_n(R)$ is not axis-reversible.

(iv) The elements e_{11} and e_{22} in $M_n(R)$ (resp. $T_n(R)$) satisfy $e_{11}e_{22} = 0$ but $e_{11}M_n(R)e_{22} = Re_{12}$ (resp. $e_{11}T_n(R)e_{22} = Re_{12}$) and $e_{22}T_n(R)e_{22} = Re_{21}$ (resp. $e_{22}T_n(R)e_{22} = 0$).

(v) It is straightforward as in (i).

(vi) It is proved by the same techniques used in proving (ii).

The next example shows that the converse of parts (i) and (v) of the previous theorem is not necessarily true.

Example 2. Let $A_1(F)$ be the first Weyl algebra over the real number field, i.e., $A_1(F)$

$= F[x, y]/\langle xy - yx - 1 \rangle$, and let R be the trivial extension of $A_1(F)$ by $A_1(F)$. Then, R is axis-reversible by Theorem 1(i). Let S be the trivial extension of R by R . Elements of S can be represented as upper triangular matrices of the form

$$\begin{bmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}, \quad a, b, c, d \in A_1(F),$$

with multiplication induced by the trivial extension structure. Consider the following elements:

$$\alpha = \begin{bmatrix} 0 & x & yx+1 & 0 \\ 0 & 0 & 0 & yx+1 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & -1 & y & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In S . A direct computation shows that $\alpha\beta = 0$. For an arbitrary element

$$t\gamma = \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix}$$

in S , we have

$$\alpha\gamma\beta = \begin{bmatrix} 0 & x & yx+1 & 0 \\ 0 & 0 & 0 & yx+1 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{bmatrix} \begin{bmatrix} 0 & -1 & y & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & xay - xya \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, $\alpha\beta = ((0, 0), (0, x[A_1(F), y]))$, where $[\cdot, \cdot]$ denotes the additive commutator in $A_1(F)$. Similarly, one can show that $\beta\alpha = ((0, 0), (0, [A_1(F)x, y]))$. Therefore, $\alpha\beta \neq \beta\alpha$, and hence S is not axis-reversible.

The next proposition provides an equivalent condition of axis-reversibility.

Proposition 1. A ring R is axis-reversible if and only if R satisfies the condition:

(†) For any nonempty subsets $A, B \subseteq R$ with $AB = 0$, we have $ARB = BRA$.

Proof. Suppose that R is an axis-reversible ring, and let $A, B \subseteq R$ be nonempty

subsets such that $AB = 0$. Then, $ab = 0$ for all $a \in A$ and $b \in B$, so $aRb = bRa$ by the definition of axis-reversibility. It follows that $ARB = \sum_{a \in A, b \in B} aRb = \sum_{a \in A, b \in B} bRa = BRA$, as required.

Conversely, the statement is clear by taking $A = \{a\}$ and $B = \{b\}$ for arbitrary $a, b \in R$ with $ab = 0$. $\square$

Next, we present results illustrating the relation between axis-commutativity and other ring-theoretic properties. Let us start with the following result.

Proposition 2. Every axis-reversible ring is 2-primal.

Proof. Let R be an axis-reversible and $a \in N(R)$ with index n . So, $anr = 0$ for every $r \in R$ and $an-1 Rar = arRan-1$ which implies $an-1 RaRa = 0$. Now, we have $(ra)(an-2sata) = 0$, for every $r, s, t \in R$. Hence, $(ra)R(an-2sata) = (an-2sata)R(ra)$. Multiplying the last equation by a from the left, we have $a(ra)R(an-2sata) = a(an-2sata)R(ra) \in an-1 RaRaR = 0$. Hence, $aRaRan-2 RaRa = 0$. Continuing, we obtain that $(aR)^{n-2} = 0$ and R is 2-primal. $\square$

Notice that the converse of the previous proposition does not necessarily hold as in the rings of the following example.

Example 3. Let $B = \mathbb{Z}_2[x]/\langle x^2 \rangle$ and $M = xB$, and consider the ring $R = \begin{bmatrix} B & M \\ M & B \end{bmatrix}$. Since B is a commutative local ring with $N(B) = \langle x \rangle$, it follows that $N(B)$ is prime and hence B is 2-primal. Indeed, $\mathcal{P}(R) = \begin{bmatrix} 0 & M \\ M & 0 \end{bmatrix}$ and $N(R) = \mathcal{P}(R)$. Therefore, R is a 2-primal ring. Consider the element

$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have $ab = 0$ while $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} = aRb \neq bRa = \begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix}$. Thus, R is not axis-reversible.

It is well known that every reduced (resp. domain) ring R is semiprime (resp. prime).

The following theorem shows that the converse holds provided R is axis-semiprime.

Theorem 2. For any ring R , the following statements hold:

- (i) R is reduced if and only if R is semiprime and axis-reversible.
- (ii) R is a domain if and only if R is prime and axis-reversible.

Proof. (i) $(\Rightarrow)$ Suppose $ab = 0$ for some $a, b \in R$. For any $r \in R$, we have $(bra)^2 = 0$, and hence $bRa = 0$ by the reducedness of R . In particular, $ba = 0$, and repeating the same argument yields $aRb = 0$. Thus, R is axis-reversible.

$(\Leftarrow)$ This follows directly from Proposition 2.

- (ii) $(\Rightarrow)$ This is immediate.

($\Leftarrow$) Suppose $ab = 0$ for some nonzero $a, b \in R$. Since R is axis-reversible, we have $aRb = bRa$. Multiplying on the right by b gives $aRb^2 = 0$. By the primeness of R , it follows that $b^2 = 0$, and hence $b = 0$ because R is reduced by part (i).

Remark 1. We consider the following conditions:

(*) If $IJ = 0$ for all right (or left) ideals I, J of R , then $IRJ = JRI$. (**) If $IJ = 0$ for all (two-sided) ideals I, J of R , then $IRJ = JRI$.

Note that $(\dagger) \Rightarrow (*) \Rightarrow (**)$, and thus, a ring R is reflexive if and only if it satisfies $(*)$, if and only if it satisfies $(**)$, by ([13] [Lemma 2.1]). Therefore, every axis-reversible ring is reflexive, but the converse fails by ([13] [Theorem 2.6(2)]) and Theorem 1(iv). We include an additional counterexample below.

Example 4. Let S be a nonzero reduced ring and n a positive integer. Define R_n to be the $2n \times 2n$

upper triangular matrix ring over S , and define the map $\sigma : R_n \rightarrow R_{n+1}$ by

$$\sigma(A) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}.$$

Consider the ring R as the direct limit of the direct system (R_n, σ^{i-j}) for $i \geq j$ over $\{1, 2, \dots\}$. By ([14] [Example 2.3(3)]), R is reflexive. On the other hand, R is semiprime by ([15] [Proposition 3]), but not reduced. Therefore, by Theorem 2, R is not axis-reversible.

According to ([12] [Lemma 1.4]), every reversible ring is semicommutative. However, semicommutativity alone is not sufficient to ensure that a ring is axis-reversible, as illustrated by the two rings in the following examples.

Example 5. Let D be a commutative domain of characteristic zero, and put $R = D \oplus D$. Define the exchange automorphism $\sigma : R \rightarrow R$ by $\sigma(a, b) = (b, a)$. Then, the Nagata extension of R by R and σ , denoted N , is semicommutative. However, consider the elements $a = (0, 1), (0, 1)$ and $b = (1, 0), (0, 1)$ in N . We have $ab = 0$, while $aNb = (0, 0), (D, 0)$ and $bNa = (0, 0), (0, D)$. Thus, $aNb \neq bNa$, and hence, N is not axis-reversible.

Example 6. For any reduced ring R , the ring $S_3(R)$ is semicommutative, by ([12] [Proposition 1.2]). However, R is not axis-reversible, as shown in Theorem 1.

Despite the independence of semicommutativity and axis-reversibility, the two classes of semicommutative rings and axis-reversible rings are neither disjoint nor complementary. Indeed, there exist rings that are both semicommutative and axis-reversible, such as commutative rings, and there are rings that are neither semicommutative nor axis-reversible, such as $M_2(R)$ for any ring R . Moreover, the next proposition shows that the intersection of the two classes is precisely the class of reversible rings. First, we give the following lemma, which can be proved directly from ([13] [Proposition 2.2]).

Lemma 1. A ring R is reversible if and only if, for all $a, b \in R$, the condition $ab = 0$ implies $aRb = bRa = 0$.

Proposition 3. A ring R is reversible if and only if R is both semicommutative and axis-reversible.

Proof. For the necessity, assume $ab = 0$. Then, by the previous lemma, we have

$aRb = bRa = 0$, which means that R is both semicommutative and axis-reversible.

For the sufficiency, suppose R is semicommutative and axis-reversible. If $ab = 0$, then $aRb = 0$ by semicommutativity, and $aRb = bRa$ by axis-reversibility. Thus $bRa = 0$ as well, and the condition of the previous lemma is satisfied. $\square$

Recall that there exist semiprime rings that are not semicommutative. For example, the prime rings $M_n(\mathbb{Z}_p)$ for some integer $n > 1$ and prime p with $p \leq n$. However, the semiprime property is a sufficient condition to make an axis-reversible ring reversible.

Proposition 4. Let R be a semiprime ring. Then R is reversible if and only if it is axis-reversible.

Proof. The sufficiency follows from Proposition 2. For the necessity, assume that R is axis-reversible and that $ab = 0$ for some $a, b \in R$. Then $aRb = bRa$. Multiplying on the right by b yields $aRb^2 = 0$. For every $r \in R$, we have $(b^2ra^2)R(b^2ra^2) = (b^2ra)(aRb^2)(ra^2) = 0$, and hence $b^2ra^2 = 0$ since R is semiprime. Thus, $0 = b^2Ra^2 = b(bRa)a = b(aRb)a = (ba)R(ba)$, which implies $ba = 0$. Therefore, R is reversible.

Recall from [16] that a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all

$a, b, c \in R$. A ring R is called Armendariz if, whenever

$f(x) = \sum_{i=0}^m a_i x^i$ and $g(x) = \sum_{j=0}^n b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$, it follows that $a_i b_j = 0$ for all i and j (see [17] for details).

Using the previous proposition and ([12] [Lemma 2.7]), we extend ([18] [Proposition 15]) in the following corollary.

Corollary 1. Let R be a regular ring and suppose that there exists the classical right quotient ring Q of R . Then the following statements are equivalent:

- (i) R is Armendariz;
- (ii) R is reduced;
- (iii) R is symmetric;
- (iv) R is reversible;
- (v) R is semicommutative;
- (vi) R is axis-reversible;
- (vii) Q is Armendariz;
- (viii) Q is reduced;
- (ix) Q is semicommutative;

- (x) Q is symmetric;
- (xi) Q is reversible;
- (xii) Q is axis-reversible.

Proposition 5. Every axis-reversible ring is abelian.

Proof. Let e be an idempotent of an axis-reversible ring R . Since $e(1 - e) = 0$, we have $eR(1 - e) = (1 - e)Re = 0$. Multiplying these equalities on the left and right by e , we obtain $eR(1 - e) = 0$ and $(1 - e)Re = 0$, which together imply that e is central. Hence, every idempotent of R is central, and therefore, R is abelian.

The converse of the previous result is not always true, as in the ring of the next example.

Example 7. Consider the ring $R = S_3(D)$ for some domain D . R is abelian since it has only trivial idempotents; however, R is not axis-reversible where the elements

$$\alpha = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In [19], a generalization of semicommutative rings was introduced. A ring R is called weakly semicommutative if for any $a, b \in R$, the condition $ab = 0$ implies $aRb \subseteq N(R)$.

Proposition 6. Every axis-reversible ring is weakly semicommutative.

Proof. Let R be an axis-reversible ring, and assume $ab = 0$ for some $a, b \in R$. Since R is axis-reversible, we have $aRb = bRa$. Now $(aRb)^2 = (bRa)^2 = bR(ab)Ra = 0$, which shows $aRb \subseteq N(R)$. Hence, R is weakly semicommutative. $\square$

The converse of the previous proposition does not hold, in general. The ring N in Example 5 is not axis-reversible. However, if $ab = 0$ for some $a, b \in N$, then $aNb \subseteq (0, R)$ and $(aNb)^2 = 0$, which means that N is weakly semicommutative. Also, for every reduced ring D , the ring $S_4(D)$ is weakly semicommutative, according to ([19] [Example 2.1]), while it is not axis-reversible, as shown in Theorem 1.

Recall [20], a ring is called duo if every one-sided ideal is two-sided; equivalently, $aR = Ra$ for every $a \in R$. The following two examples show the independence of the duo and axis-reversible properties.

Example 8. Let $R = \mathbb{Z}_2\langle x, y \rangle$ be the free algebra generated by noncommuting indeterminates x and y over $\mathbb{Z}_2$ and I are the ideal of R generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$. In the same manner as in ([11] [Example 2.3]) and ([21] [Example 2]), the ring R/I is duo but not axis-reversible.

Example 9. Consider the first Weyl algebra, $A_1(F)$, defined in Example 2. As noted there, $A_1(F)$ is axis-reversible. However, $A_1(F)$ is not a duo ring. $A_1(F)$ is a simple ring and therefore has no nonzero two-sided ideals, but it does possess nontrivial left and right ideals arising from its structure as a differential operator ring. Consequently, some

one-sided ideals of $A_1(F)$ are not two-sided, and thus $A_1(F)$ fails to be duo.

The following diagram summarizes the nontrivial implications between the ring properties studied in this paper. These implications are established by Propositions 2, 3, 5, and 6, together with known results from the literature ([13] [Lemma 2.1]), ([11] [Proposition 2.7(1)]), and ([12] [Lemma 1.4]). Also, the counterexamples, including Examples 4,

3, and 5, with those in ([12] [Example 1.5]), ([22] [Example 8]), and ([23] [Example 1.1]), show that these implications are strict. The following diagram summarizes the nontrivial implications between the ring properties studied in this paper (see Figure 1).

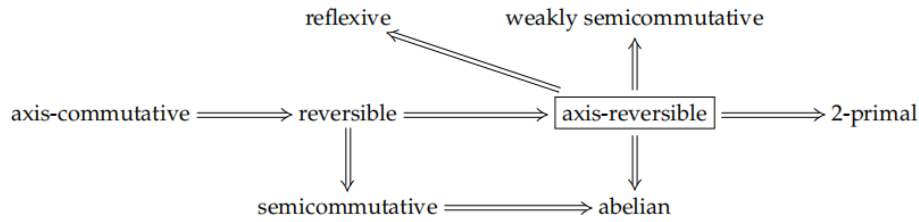


Figure 1. Summary of the nontrivial implications between the ring properties studied in this paper.

2. Extending of Axis-Reversibility

In this section, we investigate the behavior of axis-reversibility under several standard ring-theoretic constructions. We first observe that axis-reversibility is not preserved when passing through substructures, in general. In particular, this property is not stable under natural extensions or quotient constructions. The following example shows that the class of axis-reversible rings is not closed under homomorphic images.

Example 10. Consider the ring S in Example 1. Then, S is reduced and hence axis-reversible, since every reduced ring is reversible. Let I be the ideal of S generated by aSb . Then, $ab \in I$ and $aSb \subseteq I$, while $bSa \not\subseteq I$. Consequently, the factor ring S/I fails to be axis-reversible.

Remark 2. Examples 5 and 10 show that axis-reversibility is not stable under natural algebraic constructions such as Nagata-type extensions and homomorphic images. In particular, axis-reversibility is not a hereditary property, in general.

Proposition 7. Every biideal with an identity of an axis-reversible ring is also axis-reversible.

Proof. Let S be a biideal of an axis-reversible ring R with two-sided identity 1_S . Assume that $ab = 0$ for some $a, b \in S$. Since R is axis-reversible, we have $aRb = bRa$. Hence, $aSb \subseteq aRb = bRa = (b1_S)R(1_Sa) = b(1_S R 1_S)a \subseteq bSa$. Similarly, $bSa \subseteq aSb$, and therefore, $aSb = bSa$, which means that S is axis-reversible. $\square$

As an immediate consequence we obtain the following.

Corollary 2. Let R be an axis-reversible ring. Then, the corner eRe for every idempotent e of R is also axis-reversible.

The next example shows that even if every corner eRe of a ring R is axis-reversible for all non-identity idempotents e , the ring R itself is not necessarily axis-reversible.

Example 11. Let $R = T_2(\mathbb{Z})$ be the ring of all 2×2 upper triangular matrices with integer entries.

Any nontrivial idempotent e of R takes one of the two forms $\begin{bmatrix} 1 & m \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & m \\ 0 & 1 \end{bmatrix}$ which in

both cases satisfies $eRe = \mathbb{Z}$. Clearly, $\mathbb{Z}$ is commutative and, in particular, axis-reversible. However,

R itself is not axis-reversible, since $e_{11}e_{22} = 0$ while

$$0 = e_{22}Re_{11} \neq e_{11}Re_{22} = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}.$$

Now, we are showing that the class of axis-reversible rings is closed under the direct product.

Proposition 8. Let $\{R_i\}_{i \in \Lambda}$ be a family of rings for some index Λ . Then, R_i is axis-reversible, for every $i \in \Lambda$, if and only if the direct product $R = \prod_{i \in \Lambda} R_i$ is axis-reversible.

Proof. ($\Rightarrow$) Let $a = (a_i)$ and $b = (b_i)$ be elements of R satisfying $ab = 0$. So $a_i b_i = 0$ in the axis-reversible R_i , for every $i \in \Lambda$. Hence, $a_i R_i b_i = b_i R_i a_i$, for all $i \in \Lambda$. Therefore, we get $aRb = bRa$, and R is axis-reversible. ($\Leftarrow$) Let $\alpha, \beta \in R_j$, for arbitrary $j \in \Lambda$ such that $\alpha\beta = 0$. Define the two sequences $a = (a_i)$ and $b = (b_i)$ in R as $a_j = \alpha$ and $b_j = \beta$ while $a_i = b_i = 0$ for every $j \neq i \in \Lambda$. So that $ab = 0$ and consequently $aRb = 0$ since R is axis-reversible. Thus $\alpha R_j \beta = \beta R_j \alpha$, and R_j is axis-reversible. $\square$

From the Peirce Decomposition Theorem [1] together with Corollary 2, we obtain the following corollary.

Corollary 3. Let R be a ring and let e be a central idempotent of R . Then, R is axis-reversible if and only if both eR and $(1 - e)R$ are axis-reversible.

The next result extends ([24] [Proposition 6]).

Proposition 9. Let R be a commutative ring and G a finite group. Then the following statements are equivalent:

- (vii) RG is reversible;
- (viii) RG is semicommutative;
- (ix) RG is axis-reversible.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is precisely ([24] [Proposition 6]). Also, (i) $\Rightarrow$ (iii)

follows directly from Proposition 3. It remains to show that (iii) $\Rightarrow$ (i). Assume RG is axis-reversible. Let $a, b \in RG$ with $ab = 0$. By the axis-reversible property, we have $a(RG)b = (RG)ba$. Now consider the trace map $\text{tr} : RG \rightarrow R$. For every $r \in R$, there exists $s \in R$ such that $\text{tr}(rba) = \text{tr}(arb) = \text{tr}(bsa) = \text{tr}(sab) = 0$, since $ab = 0$. Thus, every element of the left ideal Rba has trace zero, which forces $ba = 0$. Hence, RG is reversible. $\square$

3. On Minimal Axis-Reversible Rings

In this section, we investigate the structure of minimal axis-reversible rings. Our aim is to identify rings of minimal order that are axis-reversible but do not satisfy any stronger structural conditions such as commutativity. In addition, we describe the structure of minimal rings that satisfy weaker conditions yet fail to be axis-reversible. Recall [1], a ring R is called semiperfect if R is semilocal and idempotents can be lifted modulo $J(R)$.

Proposition 10. A ring R is axis-reversible and semiperfect if and only if R is a finite direct sum of local axis-reversible rings.

Proof. ($\Rightarrow$) Since R is semiperfect, it has a finite orthogonal set $\{e_1, e_2, \dots, e_n\}$ of local idempotents whose sum is 1 by ([25] [Corollary 3.7.2]). Moreover, $e_i R e_i$ is a local ring for every i , by ([25] [Corollary 3.7.1]). By Corollary 2, each $e_i R e_i$ is also axis-reversible. Hence, R is a finite direct sum of local axis-reversible rings. ($\Leftarrow$) By ([25] [Corollary 3.7.1]), every local ring is semiperfect. Therefore, any finite direct sum of local axis-reversible rings is semiperfect, and axis-reversibility follows from Proposition 8. $\square$

Example 12. Let $GF(4)$ be the Galois field of order 4 and define the automorphism $\sigma : GF(4) \rightarrow GF(4)$ by $\sigma(a) = a^2$, for every $a \in GF(4)$. Then, the Nagata extension of $GF(4)$ by $GF(4)$ and σ , denoted R , is a noncommutative ring of order 16. Then $J(R) = (0, GF(4))$ and $R/J(R) \cong GF(4)$. Hence, if $xy = 0$, for nonzero elements x and y in R , then $x, y \in J(R)$. So that $xRy = yRx = 0$ and R is axis-reversible.

The next theorem demonstrates that the example above represents a minimal axis-reversible ring that is not commutative.

Theorem 3. If R is a minimal noncommutative axis-reversible ring, then R has order 16 and is isomorphic to the ring in Example 12.

Proof. By the results of [26], any noncommutative ring of minimal order is isomorphic to $T_2(Z_2)$. However, Theorem 1 shows that $T_2(Z_2)$ is not axis-reversible. Consequently, the order of R must be at least 16. Moreover, according to [26], if a finite ring admits a cube-free factorization, then it must be commutative. Since the ring in Example 12 is both noncommutative and axis-reversible, we deduce that the order of R must be 16. Now, let R be a minimal noncommutative axis-reversible ring. By Proposition 10, such a ring must be local. Therefore, $|J(R)| \in \{2, 4, 8\}$. Note that $R/J(R)$ is a field and that $J(R)$

is a vector space over $R/J(R)$. If $|J(R)| = 8$, then $R/J(R) \cong Z_2$ and $J(R)$ has a basis $\{a, b, c \mid a^2 = ab = c, ba = 0\}$ (see ([27] Theorem 2.3.6)). But in this case $aRb \neq bRa$, so R cannot be axis-reversible. If $|J(R)| = 2$, then $R/J(R) \cong GF(2)$, which forces $|J(R)| \geq 8$, a contradiction. Thus, the only possibility is $|J(R)| = 4$, in which case R is isomorphic to the ring of Example 12, as proved in ([28] [Theorem 2.6]). $\square$

With ([29] [Theorem 3]) and ([28] [Theorem 2.6]), we get directly the following corollary.

Corollary 4. For a noncommutative ring R , the following conditions are equivalent.

- (x) R is a symmetric ring of minimal order;
- (xi) R is a reversible ring of minimal order;
- (xii) R is an axis-reversible ring of minimal order.

Proposition 11. If R is a minimal non-axis-commutative axis-reversible ring, then R is of order 16

and is isomorphic to the ring of Example 12.

Proof. The result follows immediately from the arguments in [21] together with ([11] [Lemma 2.4(2)]).

4. Conclusion

This paper examines axis-reversible rings, providing a novel perspective on their construction and behavior. The introduction of axis-reversibility enhances the comprehension of ring theory, particularly within the realm of non-commutative algebra. The findings indicate that axis-reversibility is observable in many ring types, including matrix and triangular matrix rings, under particular conditions. The research delineates the relation between axis-reversibility and other significant ring attributes such as semiprimeness, reducedness, and reversibility. We illustrate through rigorous examples and arguments that axis-reversible rings are essential in connecting various algebraic structures. This study enhances the theory of non-commutative rings and facilitates further research into their applications in several domains of mathematics and theoretical physics. The findings described herein unveil new opportunities for exploring the algebraic features of rings, perhaps yielding profound insights into their wider consequences.

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