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Cubic B-spline Orthogonal collocation on finite elements method for space-time fractional KdV equation

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Abstract:

This work extends cubic B-spline orthogonal collocation (OCFE) method to space, time and space-time fractional partial differential equations with KdV equation as a case study. We incorporate the θ method in time and cubic B-spline collocation method in space using Gauss points as collocation points. The method is unconditionally stable and gives approximate results that agree closely with the exact solutions.

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1. Introduction

The numerical solution of the classical Korteweg-de Vries (KdV) equation have been sought via various methods in the literature. Some of these methods have been extended to solve the fractional case of the equation e.g Adomian decomposition method, spectral method and so on. The fractional Korteweg-de Vries (KdV) equation in space and time was solved in Momani (2005) using the Adomian de- composition method. A modified reduced fractional differential transform method was used to obtain the semi-analytic solution of time-fractional KdV equation in Saha Ray (2013). Space and time fractional differential equations were solved by variational iteration method in Momani et al. (2008). In Tauseef Mohyud-Din et al. (2012), fractional KdV equation with space and time fractional order were solved numerically with homotopy analysis method. The authors in Appadu & Kelil (2023) used the standard finite difference to obtain the numerical solution of time-fractional KdV equation. Other methods used are homotopy perturbation method (HPM) in Wang (2007), residual power series method based on generalized Taylors series in El-Ajou et al. (2015), spectral collocation method in Khader et al. (2021) and linear B-spline functions in Lakestani et al. (2012).

A numerical scheme for the solution of the time fractional Kortwege-de-Vries equation was developed in Zhang et al. (2017). The authors in Lu et al. (2020) used the fractional Elzaki projected differential transform method to obtain the numerical solution of the coupled fractional nonlinear KdV equation. The numerical solution of the fractional KdV-Burgers' equation was obtained in Wang (2006) using the Adomian decomposition method. In Han et al. (2021), the authors applied Fourier spectral method to solve space fractional, variable coefficient and modified KdV equations. (Adetona, 2024) applied quadratic and cubic B-splines to solve differential equations.

To the best of the authors' knowledge there are very few studies directed towards developing numerical schemes for solving the space-time fractional KdV equation. In this paper we develop the Orthogonal Collocation on Finite Elements (OCFE) method for solving the space-time fractional KdV equation. The OCFE method was popular in the engineering community in the seventies Finlayson (1974), Carey & Finlayson (1975)) and is particularly suited for problems whose solution exhibit steep solution profiles. In contrast to OSC (Orthogonal Spline Collocation) which solves the problem globally over the space domain OCFE method divides the domain into sub-domains (elements) and applies orthogonal collocation on each element. Naturally, this leads to a more robust way to handle problems posed on complex domains and having steep solution profiles. It is well known using OSC over the entire domain leads to oscillations in the numerical solution when high order methods are applied. We further emphasize that OCFE method is computationally efficient due to the following:

- The use of splines results in sparse systems and the application of fast matrix solvers,
- applying orthogonal collocation yields greater accuracy in the numerical approximation,
- using finite elements allows greater flexibility which leads to more stable methods.

The above features are usually present and useful to consider when solving time and space fractional PDEs.

In this work, we use the cubic B-spline orthogonal collocation on finite element (OCFE) to obtain the numerical solutions of the space-time fractional KdV equation

$$\frac{\partial^\beta u}{\partial t^\beta} = -\epsilon u \frac{\partial^\alpha u}{\partial x^\alpha} - \mu \frac{\partial^3 u}{\partial x^3} + g(x,t), \quad 0 < \alpha < 1, \quad 0 < \beta < 1, \tag{1}$$

where $u = u\ (x, t)$, n and ` are constants and 6 (x, t) is the source term. We also consider the space fractional case (β= 1) and time fractional case (α= 1). This work is organized as follows: section 1 introduces the KdV equation, section 2 describe the cubic B-spline OCFE method, section 3 deals with discretization of the cubic B-spline OCFE for space fractional partial differential equation of order $0 < \alpha < 1$, section 4 establishes the stability of the method, section 5 describes the discretization of the cubic OCFE for time fractional partial differential equation method of order $0 < \alpha < 1$, section 6 illustrates our method with examples and section 7 concludes the work.

2. Derivation of the cubic B-spline OCFE method

Consider solving a partial differential equation of the form

$$F(x, t, u, u_x, u_{xx}, u_{xxx}, u_t) = f(x,t), \quad x \in [a, b] \tag{2}$$

for $u(x, t)$, subject to appropriate boundary conditions and an initial condition. We apply collocation in the spatial variable.

$$x_l = a + (l-1)h, \quad l = 1, 2, \ldots, N,$$

Let the interval $[a, b]$ be partitioned into N finite elements with nodes given by
where h is the uniform interval spacing. For every interval $[x_i, x_{i+1}]$, we transform the variable

$$z = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{h}.$$

x to $z \in [0, 1]$ by letting

T

$$\begin{aligned} u^i(x,t) &= u^i(z,t), \\ &= \sum_{j=1}^{4} p_j^i(t) B_j(z), \end{aligned} \tag{3}$$

he trial solution on the interval is denoted by

$$B_j(z) = \binom{3}{j-1}(1-z)^{4-j}z^{j-1}, \quad j = 1, 2, 3, 4. \tag{4}$$

$$\left.\frac{\partial^k u^i(x,t)}{\partial x^k}\right|_{x_{k+1}} = \left.\frac{\partial^k u^{i+1}(x,t)}{\partial x^k}\right|_{x_{k+1}}, \quad k = 0, 1, 2. \tag{5}$$

$$p_1^{i+1} - p_4^i = 0, \tag{6}$$

$$p_2^{i+1} - 2p_4^i + p_3^i = 0, \tag{7}$$

$$p_3^{i+1} - p_2^i + 4p_3^i - 4p_4^i = 0, \tag{8}$$

$$i = 2, 3, \ldots, N. \tag{9}$$

Where

$$u(z,t) = \sum_{k=1}^{4} p_{k+3(i-1)}(t)B_k(z). \tag{10}$$

The boundary condition on the i_{th} and $(i + 1)_{th}$ interval are given by
This leads to the equations

Using the continuity equation (6) we may rewrite the trial solution on the i_{th} interval as
Using a single Gauss collocation point per interval, three boundary conditions and $2(N-1)$ continuity equations from (7) and (8), we obtain a system of $3N + 1$ equations in $3N + 1$ unknowns. The solution of this system yields the coefficient function $p_j(t)$, $j = 1, 2 \ldots 3N + 1$.
The $D_x^\alpha u(x) = \frac{d^\alpha}{dx^\alpha}u(x)$ solution at the node x_i is given by $(x_i, t) = p_{3i-2}(t)$.

3. Discretization of the space fractional differential equation of order $0 < \alpha < 1$

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$$D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)}\int_{x_1}^{x}(x-\xi)^{-\alpha}u'(\xi)d\xi, \quad 0 < \alpha < 1, \tag{11}$$

$$x \in [x_i, x_{i+1}].$$

Let be the space fractional derivative of order α, $0 < \alpha < 1$.

$$D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)}\sum_{j=1}^{i-1}\int_{x_j}^{x_{j+1}}(x-\xi)^{-\alpha}\sum_{k=1}^{4}p_{k+3(j-1)}B_k'(\xi)\,d\xi + \frac{1}{\Gamma(1-\alpha)}\int_{x_i}^{x}(x-\xi)^{-\alpha}\sum_{k=1}^{4}p_{k+3(i-1)}B_k'(\xi)\,d\xi.$$

$$D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=1}^{i-1}\sum_{k=1}^{4}p_{k+3(j-1)}\int_{x_j}^{x_{j+1}}B_k'(\xi)(x-\xi)^{-\alpha}d\xi + \sum_{k=1}^{4}p_{k+3(i-1)}\int_{x_i}^{x}(x-\xi)^{-\alpha}B_k'(\xi)\,d\xi\right],$$

We use the Caputo fractional derivative for the space fractional derivative of order α given by

$$B_1(\xi) = \left(1 - \frac{\xi - x_j}{h}\right)^3, \quad B_2(\xi) = 3\left(\frac{\xi - x_j}{h}\right)\left(1 - \frac{\xi - x_j}{h}\right)^2,$$

We substitute (10) in (11) and obtain
where

$$B_3(\xi) = 3\left(\frac{\xi - x_j}{h}\right)^2\left(1 - \frac{\xi - x_j}{h}\right), \quad B_4(\xi) = \left(\frac{\xi - x_j}{h}\right)^3,$$

and $h = x_{j+1} - x_j$.

The fractional derivative can be written in the form

$$D_x^\alpha u(x) = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=1}^{i-1}\left[p_{3j-2}J_1 + p_{3j-1}J_2 + p_{3j}J_3 + p_{3j+1}J_4\right] + p_{3i-2}I_1 + p_{3i-1}I_2 + p_{3i}I_3 + p_{3i+1}I_4\right],$$

where

$$J_k = \int_{x_j}^{x_{j+1}} B_k'(\xi)(x-\xi)^{-\alpha}\, d\xi,$$
$$I_k = \int_{x_i}^{x} B_k'(\xi)(x-\xi)^{-\alpha}\, d\xi, \quad k = 1, 2, 3, 4.$$

Since $x = x_i + zh$ where $i = 1, 2, \ldots, N.$,

$$x - x_j = (i + z - j)h,$$

and

$$x - x_{j+1} = (i + z - j - 1)h.$$

Hence

$$\begin{aligned}
&D_x^\alpha u(x_i + hz) = \\
&\frac{-3h^{-\alpha}}{\Gamma(4-\alpha)}\left[\sum_{j=1}^{i-1}\sum_{k=1}^{4}\left[p_{3j-2}\left[-(\alpha-2)(\alpha-3)\left((i+z-j-1)^{1-\alpha} - (i+z-j)^{1-\alpha}\right)\right.\right.\right.\\
&+2\left((i+z-j-1)^{1-\alpha}(\alpha-1-i-z+j) + (i+z-j)^{2-\alpha}\right)(\alpha-3)\\
&-\left(\alpha^2 - \alpha\,(3+2i+2z-2j) + 6i + 6z - 6j + 2\,(i+z-j-1)^2\right)\times(i+z-j-1)^{1-\alpha}\\
&+2\,(i+z-j)^{3-\alpha}\big] + p_{3j-1}\left[(\alpha-2)(\alpha-3)\left((i+z-j-1)^{1-\alpha} - (i+z-j)^{1-\alpha}\right)\right.\\
&-4\left((i+z-j-1)^{1-\alpha}(\alpha-1-i-z+j) + (i+z-j)^{2-\alpha}\right)(\alpha-3)\\
&+3\left(\alpha^2 - \alpha\,(3+2i+2z-2j) + 6i + 6z - 6j + 2\,(i+z-j-1)^2\right)(i+z-j-1)^{1-\alpha}\\
&-6\,(i+z-j)^{3-\alpha}\big] + p_{3j}\left[2\left((i+z-j-1)^{1-\alpha}(\alpha-1-i-z+j) + (i+z-j)^{2-\alpha}\right)\right.\\
&\times(\alpha-3) - 3\left(\alpha^2 - \alpha\,(3+2i+2z-2j) + 6i + 6z - 6j + 2\,(i+z-j-1)^2\right)\\
&\times(i+z-j-1)^{1-\alpha} + 6\,(i+z-j)^{3-\alpha}\big] + p_{3j+1}\left[(i+z-j-1)^{1-\alpha}\right.\\
&\times\left(\alpha^2 - \alpha\,(3+2i+2z-2j) + 6i + 6z - 6j + 2\,(i+z-j-1)^2\right)\\
&-2\,(i+z-j)^{3-\alpha}\big]\big] + \sum_{k=1}^{4}\left[p_{3i-2}\left[(\alpha-2)(\alpha-3)\,z^{1-\alpha} + 2\,(\alpha-3)\,z^{2-\alpha} + 2\,z^{3-\alpha}\right]\right.\\
&+p_{3i-1}\left[-(\alpha-2)(\alpha-3)\,z^{1-\alpha} - 4\,(\alpha-3)\,z^{2-\alpha} - 6\,z^{3-\alpha}\right]\\
&\left.+p_{3i}\left[2\,(\alpha-3)\,z^{2-\alpha} + 6\,z^{3-\alpha}\right] + 2p_{3i+1}\,z^{3-\alpha}\big]\right],\\
&i = 1, 2, 3\ldots, N.
\end{aligned} \tag{12}$$

The boundary conditions are discretized by using (10). The whole system is then solved for the unknowns $p_k, k = 1,2, \dots, 3N + 1$ and obtain the numerical solution from equation (10).

4. Stability of cubic B-spline OCFE for KdV equation

For β= 1 and $g(x,t) = 0$ equation (1) may be written in the form

$$u_t = -\epsilon u D_x^\alpha u - \mu u_{xxx}. \quad (13)$$

Integrating (13) over the interval $[t_n, t_{n+1}]$ we have

$$\int_{t_n}^{t_{n+1}} u_t \, dt = -\epsilon \int_{t_n}^{t_{n+1}} u D_x^\alpha u \, dt - \mu \int_{t_n}^{t_{n+1}} u_{xxx} \, dt. \quad (14)$$

We shall apply the trapezoidal rule on the right hand side to obtain

$$u^{n+1} - u^n = -\epsilon \frac{\Delta t}{2}\left(u^n D_x^\alpha u^n + u^{n+1} D_x^\alpha u^{n+1}\right) + \mu \frac{\Delta t}{2}\left(u_{xxx}^{n+1} + u_{xxx}^n\right). \quad (15)$$

Using the quasi-linearization technique to handle the term $u^{n+1}D_x^\alpha u^{n+1}$, we

$$u_i^{n+1} + \frac{\epsilon \Delta t}{2} u_i^{n+1} D_x^\alpha u_i^n + \frac{\epsilon \Delta t}{2} u_i^n D_x^\alpha u_i^{n+1} + \frac{\mu \Delta t}{2} u_{xxx,i}^{n+1} = u_i^n - \frac{\mu \Delta t}{2} u_{xxx,i}^n. \quad (16)$$

Let $K = \max_{[t_n, t_{n+1}]} u$ be a local constant that represents D in the nonlinear space term of equation (15) then dropring the subscript i

$$u^{n+1} + \frac{\epsilon \Delta t}{2} K D_x^\alpha u^{n+1} + \frac{\mu \Delta t}{2} u_{xxx}^{n+1} = u^n - \frac{\mu \Delta t}{2} u_{xxx}^n - \frac{\epsilon \Delta t}{2} K D_x^\alpha u^n. \quad (17)$$

$$D_x^\alpha u = \frac{-3h^{-\alpha}}{\Gamma(4-\alpha)}\left[\sum_{j=1}^{i-1}\sum_{k=1}^{4} p_{k+3(j-1)} J_k(z) + \sum_{k=1}^{4} p_{k+3(i-1)} I_k(z)\right], \quad (18)$$

$$I_1\left(\frac{1}{2}\right) = I_\alpha(-2\alpha^2 + 8\alpha - 7), \quad I_2\left(\frac{1}{2}\right) = I_\alpha(2\alpha^2 - 6\alpha + 3),$$
$$I_3\left(\frac{1}{2}\right) = I_\alpha(3 - 2\alpha), \quad I_4\left(\frac{1}{2}\right) = I_\alpha,$$

where $I_\alpha = \frac{3}{2^{2-\alpha} h^\alpha (3-\alpha)(2-\alpha)(1-\alpha)}$,

$$\left(i - j + \frac{1}{2}\right)^{1-\alpha} - \left(i - j - \frac{1}{2}\right)^{1-\alpha} =$$
$$(i-j)^{1-\alpha}\left[1 + \frac{1}{2(i-j)}\right]^{1-\alpha} - (i-j)^{1-\alpha}\left[1 - \frac{1}{2(i-j)}\right]^{1-\alpha},$$
$$\approx (i-j)^{1-\alpha}\left[1 + \frac{1-\alpha}{2(i-j)} - \left(1 - \frac{1-\alpha}{2(i-j)}\right)\right],$$
$$= (1-\alpha)(i-j)^{-\alpha}.$$

Similarly,

$$J_1\left(\frac{1}{2}\right) \approx \frac{-J_\alpha}{(i-j)^\alpha}, \quad J_2\left(\frac{1}{2}\right) \approx \frac{-J_\alpha}{(i-j)^\alpha},$$
$$J_3\left(\frac{1}{2}\right) \approx \frac{J_\alpha}{(i-j)^\alpha}, \quad J_4\left(\frac{1}{2}\right) \approx \frac{J_\alpha}{(i-j)^\alpha},$$

where $J_\alpha = \frac{3}{4h^\alpha}$.

LHS of (17) is

$$D_x^\alpha u\left(x_i + \frac{1}{2}h\right) = \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=1}^{i-1}\left[p_{3j-2} J_1\left(\frac{1}{2}\right)\right.\right.$$
$$\left.\left. + p_{3j-1} J_2\left(\frac{1}{2}\right) + p_{3j} J_3\left(\frac{1}{2}\right) + p_{3j+1} J_4\left(\frac{1}{2}\right)\right]\right] \quad (19)$$

$$+\sum_{k=1}^{4} p_{k+3(i-1)} I_k\left(\frac{1}{2}\right)\Bigg].$$

The first sum simplifies to

$$\frac{J_\alpha}{\Gamma(1-\alpha)}\left[(p_{3j+1}-p_{3j-2})+(p_{3j}-p_{3j-1})\right](i-j)^{-\alpha}. \tag{20}$$

The second sum is

$$\begin{aligned}\frac{I_\alpha}{\Gamma(1-\alpha)}&\left[p_{3i+1}+(-2\alpha^2+8\alpha-7)p_{3i-2}\right.\\ &\left.+p_{3i}(3-2\alpha)+(2\alpha^2-6\alpha+3)p_{3i-1}\right].\end{aligned} \tag{21}$$

Let

$$\begin{aligned} 1 &= A_1+\Delta_1, & 3-2\alpha &= A_2+\Delta_2,\\ -2\alpha^2+8\alpha-7 &= A_1-\Delta_1, & 2\alpha^2-6\alpha+3 &= A_2-\Delta_2.\end{aligned}$$

This gives

$$\begin{aligned} A_1 &= -\alpha^2+4\alpha-3, & A_2 &= \alpha^2-4\alpha+3,\\ \Delta_1 &= \alpha^2-4\alpha+4, & \Delta_2 &= -\alpha^2+2\alpha.\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{I_\alpha}{\Gamma(1-\alpha)}&\left[A_1(p_{3i+1}+p_{3i-2})+\Delta_1(p_{3i+1}-p_{3i-2})\right.\\ &\left.+A_2(p_{3i}-p_{3i-1})+\Delta_2(p_{3i}-p_{3i-1})\right].\end{aligned} \tag{23}$$

LHS of (17) is

$$u_i^{n+1}+a u_{xxx,i}^{n+1}+bD_x^\alpha u_i^{n+1}, \tag{24}$$

where $a=\frac{\mu\Delta t}{2}$, $b=\frac{\epsilon\Delta t k}{2}$. Let $p_m^n=\lambda^n e^{\mathrm{i}mhl}$.

$$\begin{aligned} u_i^{n+1} &= \lambda^{n+1}\sum_{k=1}^{4} e^{\mathrm{i}(k+3i-3)hl}B_k\left(\frac{1}{2}\right),\\ &= \lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}\sum_{k=1}^{4} e^{\mathrm{i}(k-\frac{5}{2})hl}B_k\left(\frac{1}{2}\right),\\ &= \frac{\lambda^{n+1}}{4}e^{\mathrm{i}(3i-\frac{1}{2})hl}\left[3\cos(w)+\cos(3w)\right],\ w=\frac{hl}{2}.\end{aligned} \tag{25}$$

$$\begin{aligned} u_{xxx,i}^{n+1} &= \lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}\sum_{k=1}^{4} e^{\mathrm{i}(k-\frac{5}{2})hl}B_k'''\left(\frac{1}{2}\right),\\ &= \lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}12\mathrm{i}\left[-3\sin(w)+\sin(3w)\right].\end{aligned} \tag{26}$$

(20) simplifies to

$$\frac{\lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}}{\Gamma(1-\alpha)}J_\alpha 2\mathrm{i}e^{-\mathrm{i}(6wi)}\sum_{j=1}^{i-1}e^{\mathrm{i}(6wj)}\left[\sin(3w)+\sin(w)\right](i-j)^{-\alpha}.$$

(21) simplifies to

$$\begin{aligned}\frac{\lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}}{\Gamma(1-\alpha)}&2I_\alpha\left[A_1\cos(3w)+A_2\cos(w)\right.\\ &\left.+\mathrm{i}(\Delta_1\sin(3w)+\Delta_2\sin(w))\right].\end{aligned} \tag{27}$$

and (24) becomes

$$\lambda^{n+1}e^{\mathrm{i}(3i-\frac{1}{2})hl}\left[T_1+T_2+T_3+T_4\right], \tag{28}$$

where

$$\begin{aligned} T_1 &= \frac{1}{4}(3\cos(w)+\cos(3w)),\\ T_2 &= c_2\mathrm{i}(-3\sin(w)+\sin(3w)),\quad c_2=I_{2a},\\ T_3 &= c_3\mathrm{i}e^{-\mathrm{i}6wi}\sum_{j=1}^{i-1}e^{\mathrm{i}6wj}\left[\sin(3w)+\sin(w)\right](i-j)^{-\alpha}, c_3=\frac{26J_\alpha}{\Gamma(1-\alpha)},\end{aligned}$$

$$T_4 = c_4 \left[A_1 \cos(3w) + A_2 \cos(w) \mathrm{i}(\Delta_1 \sin(3w) + \Delta_2 \sin(w))\right], c_4 = \frac{26 I_\alpha}{\Gamma(1-\alpha)}. \tag{29}$$

The RHS of (17) becomes

$$\lambda^n e^{\mathrm{i}(3i-\frac{1}{2})hl} \left[T_1 - T_2 - T_3 - T_4\right]. \tag{30}$$

Taking the ratio of (28) to (30) we have

$$|\lambda|^2 = \left|\frac{T_1 - T_2 - T_3 - T_4}{T_1 + T_2 + T_3 + T_4}\right|^2, = \frac{N}{D},$$

where

$$T_2 = \mathrm{i}\beta, \quad T_3 + T_4 = \delta + \mathrm{i}\gamma,$$
$$N = (T_1 - \delta)^2 + (\beta + \gamma)^2,$$
$$D = (T_1 + \delta)^2 + (\beta + \gamma)^2.$$

$$N - D = -4T_1\delta, = -(3\cos(w) + \cos(3w)) \left[c_3(\sin(3w) + \sin(w)) \times \sum_{j=1}^{i-1} \sin(6w(i-j))(i-j)^{-\alpha} + c_4 A_2(\cos(w) - \cos(3w))\right].$$

N.B: $A_1 = -A_2$.
By simple trigonometry, we can simplify

$$N - D = -16\cos^4(w) \left[c_3 \cos(w)\sin(w) \sum_{j=1}^{i-1} \sin(6w(i-j))(i-j)^{-\alpha} + c_4 A_2 S^2 \right]. \tag{31}$$

where $S = \sin(w)$. For x small we shall use the approximation $\sin(x) \approx x$ and $\cos(w) \approx 1$. The summation in (31) then can be simplified as

$$\sum_{j=1}^{i-1} \sin(6w(i-j))(i-j)^{-\alpha} \approx 6w \sum_{j=1}^{i-1} (i-j)^{1-\alpha}, \approx 6w \int_1^i (i-x)^{1-\alpha} dx, = \frac{6w(i-1)^{2-\alpha}}{2-\alpha}.$$

Hence (31) simplifies to

$$N - D = -16\cos^4(w) \left| c_3 \frac{6w^2}{2-\alpha}(i-1)^{2-\alpha} + c_4 A_2 w^2 \right| \leq 0,$$
$$\implies |\lambda|^2 \leq 1,$$
$$|\lambda| \leq 1.$$

Note that we assume that $k > 0$ so that c_3 and c_4 are positive. It can be verified that $|\lambda| \leq 1$. Therefore cubic space fractional OCFE is unconditionally stable.

A theoretical error analysis is beyond the scope of the paper. The interested reader can refer to (Adetona, 2024) for a complete error analysis in the fractional ODE case. A similar analysis may be obtained for the fractional PDE case in an analogous manner.

5. Discretization of the time fractional differential equation of order $0 < \beta < 1$

Suppose the time interval $[t_0, t_f]$ is divided into M partitions such that $t_0 = t_1 < t_2 < \ldots < t_{M+1} = t_f$ with $\Delta_f = \frac{t_f - t_0}{M}$. The discretized form of the Caputo time derivative $(0 < \beta < 1)$ is given by:

$$D_t^{\beta} u(x, t_{i+1}) = \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)} \Bigg[u(x, t_{i+1}) - u(x, t_i) + \sum_{j=1}^{i-1} [u(x, t_{j+1}) - u(x, t_j)] \times \left((i-j+1)^{1-\beta} - (i-j)^{1-\beta} \right) \Bigg], \quad i = 1, 2 \ldots, M. \tag{32}$$

When $i = 1$, we ignore the term of equation (32) that contains the summation sign.

6. Numerical examples

In this section we use the results of previous sections in the following cases of equation (1):

Example 6.1: Consider equation (1) with $\alpha = 1$, $\epsilon = 6$, $\mu = 1$, $g(x, t) = 0$. Momani et al. (2008), the initial condition

$$u(x, 0) = 0.5\text{sech}^2(0.5x), \tag{33}$$

and boundary conditions

$$u(0, t) = 0, \quad u(1, t) = 0, \quad u_x(1, t) = 0. \tag{34}$$

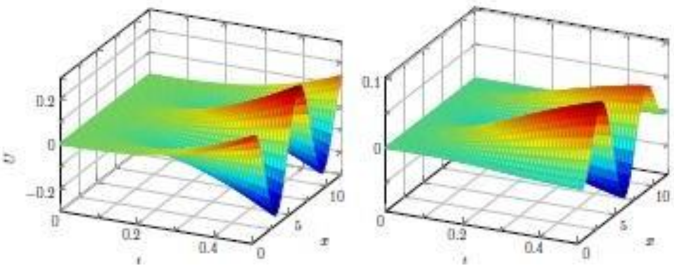
The exact solution to this problem is taken as the analytical result obtained in Momani et al. (2008). Note that when $\beta = 1$ $u(x, t) = 0.5sech^2(0.5(x - t))$.

Here we have used the right hand rectangular rule for the term on the left hand side and last term on the right hand side of (1), and trapezoidal rule for the rest which is $O(\Delta t^2)$. Further linearizing the non-linear term in time at x_i results in

$$\begin{aligned} &\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)} u_i^{n+1} + \frac{\epsilon}{2} u_i^{n+1} u_{x,i}^n + \frac{\epsilon}{2} u_i^n u_{x,i}^{n+1} + \frac{\mu}{2} u_{xxx,i}^{n+1} = \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)} u_i^n - \frac{\mu}{2} u_{xxx,i}^n \\ &- \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)} \sum_{j=1}^{n-1} \left(u_i^{j+1} - u_i^j \right) \left[(n-j+1)^{1-\beta} - (n-j)^{1-\beta} \right] \\ &+ \theta g(x_i + hz, t_{n+1}) + (1-\theta) g(x_i + hz, t_n), \\ &n = 1, 2 \ldots, Nt, \quad i = 1, 2 \ldots, N, \end{aligned} \tag{35}$$

where $0 \le \theta \le 1$. 3D plots of the solutions and errors of the numerical simulations with various values of β when $N = Nt = 50$ and $\theta = 0.5$ are displayed in Figure 1.

In Figure 1, the errors decreases as the values of β tends to 1.



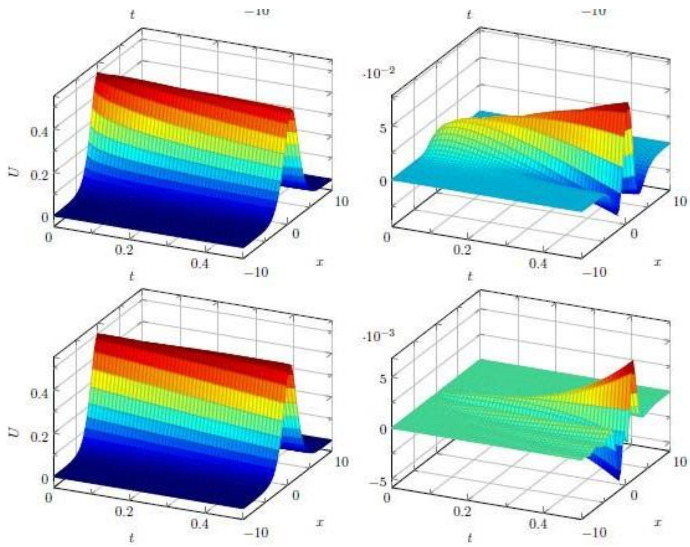


Figure 1: Numerical solution of $\beta = 0.25$ (top), $\beta = 0.5$ (middle), β = 1.0 (bottom row).

Example 6.2: Consider the equation

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} + \frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x} = g(x,t) \qquad (36)$$

with $g(x,t) = \frac{2t\cos(x)}{\Gamma(3-\alpha)}$, Momani et al. (2008), the initial condition

$$u(x,0) = 0, \qquad (37)$$

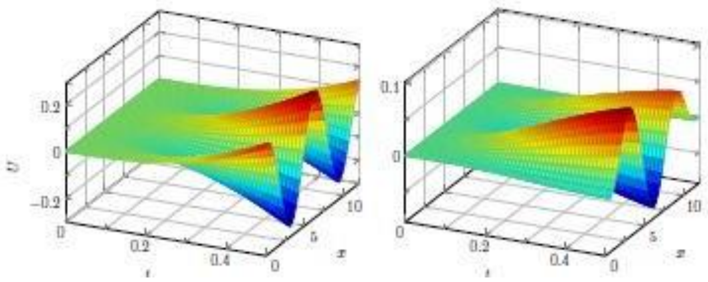
and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u_x(1,t) = 0. \qquad (38)$$

The approximate analytical solution to this problem obtained in Momani et al. (2008) is

$$u(x,t) = \left(\frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{6t^{4-2\alpha}}{\Gamma(5-2\alpha)} + \frac{2t^{5-3\alpha}}{\Gamma(6-3\alpha)} \right) \cos(x). \qquad (39)$$

The numerical solutions and errors obtained for some values of β when $N = Nt = 50$ and $\theta = 0.5$ are shown in Figure 2. In Figure 2, the errors decreases as the values of β tends to 1.



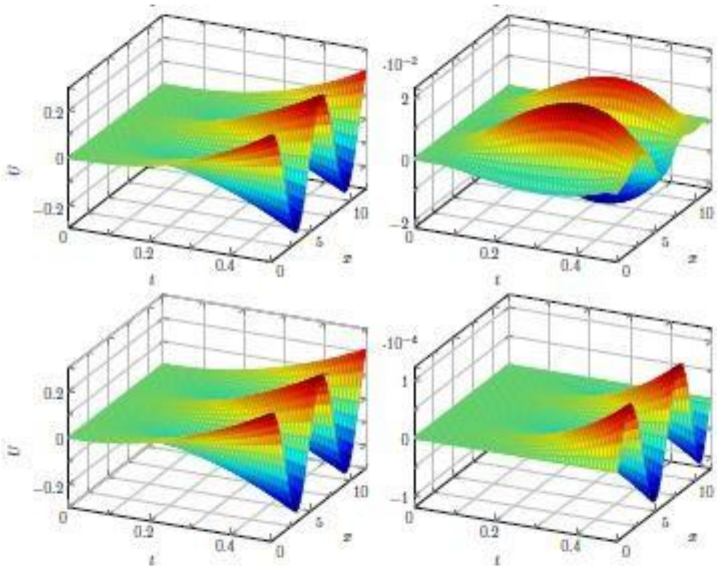


Figure 2: Numerical solution of $\beta = 0.25$ (top), $\beta = 0.5$ (middle), $\beta = 1.0$ (bottom row).

Example 6.3: Consider the space-time fractional KdV equation (1) with $\epsilon = 1$, $\mu = 1$, $g(x,t) = \frac{t^{1-\beta}x^3}{\Gamma(2-\beta)} + 6\,\frac{t^2x^{6-\alpha}}{\Gamma(4-\alpha)} + 6\,t$, the initial condition

$$u(x,0) = 0 \tag{40}$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,t) = t, \quad u_x(1,t) = 3t. \tag{41}$$

The exact solution to this problem is $u(x,t) = tx^3$. The linearized form of (1) Is

$$\begin{aligned} &\frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}u_i^{n+1} + \frac{\epsilon}{2}u_i^{n+1}D_x^{\alpha}u_i^n + \frac{\epsilon}{2}u_i^n D_x^{\alpha}u_i^{n+1} \\ &+ \frac{\mu}{2}u_{xxx,i}^{n+1} = \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}u_i^n - \frac{\mu}{2}u_{xxx,i}^n \\ &- \frac{\Delta t^{-\beta}}{\Gamma(2-\beta)}\sum_{j=1}^{n-1}\left(u_i^{j+1} - u_i^j\right)\left[(n-j+1)^{1-\beta} - (n-j)^{1-\beta}\right] \\ &+ \theta g(x_i + hz, t_{n+1}) + (1-\theta)g(x_i + hz, t_n), \\ &n = 1, 2 \ldots, Nt, \quad i = 1, 2 \ldots, N. \end{aligned} \tag{42}$$

We calculate the convergence rate using the formula $r = \frac{\log\left(e_h/e_{\frac{h}{2}}\right)}{\log 2}$

where $e = ||u(x_i, t_j) - U(x_i, t_j)||$ is the error, $u(x, t)$ and $U(x, t)$ are the exact and approximate solutions, respectively.

Figures 3 shows the approximate solution for some values of α and β. Table 1 shows the convergence rates for space fractional KdV equation i.e case β = 1 while Figure 2 shows that of the time fractional case (α = 1). The convergence rates is approximately 2 in Table 1 while it is approximately 1 in Table 2.

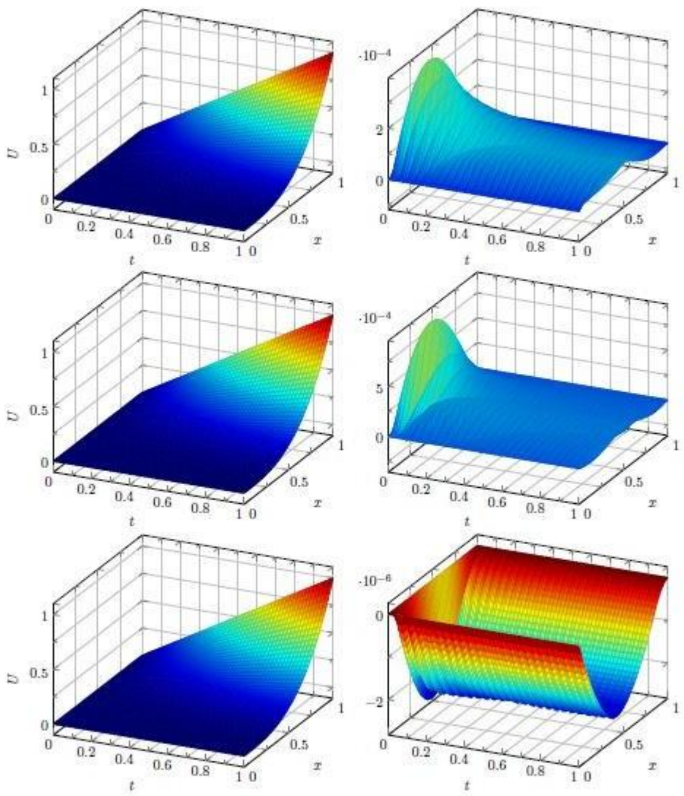


Figure 3: Numerical solution α = β = 0.25 (top), α = β = 0.5 (middle) , α = β = 1.0 (bottom row).

Table 3 displays the convergence rates for the space-time fractional case where the values of α and β are varied. In Table 3, θ = 0.5 gives the highest convergence rates and therefore gives the best approximation. Tables 4 and 5 show the infinity error norms when θ = 1 and θ = 0.5 respectively, for $\alpha, \beta \in \{0, 0.2, 0.4, 0.5, 0.6, 0.8\}$. The infinity error norm for θ= 1 in Table 4 is hundred times higher than that of θ = 0.5 in Table 5 using the same number of time and space step sizes.

Table 1: Convergence rates for different values of α when β = 1, N = 50 and Δt = 0.02.

x	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.8$
0.10	1.9559	1.9576	2.0012	1.9136
0.20	1.9757	1.9769	2.0168	1.9271
0.30	1.9982	1.9986	2.0329	1.9421
0.40	2.0217	2.0207	2.0476	1.9543
0.50	2.0057	2.0051	2.0216	1.9281
0.60	1.9720	1.9732	1.9755	1.8838
0.70	2.0117	2.0114	1.9938	1.8983
0.80	2.0332	2.0317	1.9892	1.8825
0.90	1.9092	1.9119	1.8313	1.7026
0.96	1.8465	1.8513	1.7216	1.6303

x	$\beta = 0.2$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.8$
0.10	1.1121	0.9264	0.9796	1.0076
0.20	1.0844	0.9572	0.9728	1.0124
0.30	1.0500	0.9908	0.9784	1.0062
0.40	1.0114	1.0180	0.9970	0.9946
0.50	0.9725	1.0276	1.0173	0.9926
0.60	0.9378	1.0139	1.0217	1.0060
0.70	0.9116	0.9821	1.0012	1.0149
0.80	0.8972	0.9478	0.9657	0.9941
0.90	0.8955	0.9281	0.9395	0.9587
0.96	0.8992	0.9267	0.9358	0.9494

Table 2: Convergence rates for different values of β when α = 1, N = 50 and Δt = 0.02.

Table 3: Convergence rates for different values of θ when N = 50, α = β = 0.5 and Δt =0.02.

t	$\theta = 0$	$\theta = \frac{1}{2}$	$\theta = \frac{1}{3}$	$\theta = \frac{1}{4}$	$\theta = \frac{1}{5}$	$\theta = 1$
0.10	0.7126	0.1129	0.5024	0.6374	0.6152	0.9047
0.36	1.0408	1.3419	1.0538	1.1176	1.0183	1.0074
0.40	1.0819	2.1262	1.1210	1.1780	1.0693	1.0208
0.50	1.1200	2.1926	1.1782	1.2355	1.1155	1.0396
0.60	1.0514	1.2453	1.0626	1.1476	1.0321	1.0341
0.68	0.9408	0.2676	0.8837	1.0072	0.8997	1.0140
0.80	0.7605	1.3726	0.5622	0.7773	0.6746	0.9736
0.86	0.7004	2.5340	0.3809	0.7048	0.5782	0.9558
0.90	0.6886	1.6208	0.2225	0.6971	0.5299	0.9477
0.92	0.6971	1.3947	0.0941	0.7151	0.5212	0.9475

Table 4: Infinity error norms ($L_\infty \times 10^3$) for different values of α and β when θ = 1, N =50 and Δt=0.02.

β	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.8$
0	5.9301	5.9119	5.9340	5.8801	5.8766	5.8559
0.2	8.4593	8.4255	8.4212	8.4057	8.3988	8.3017
0.4	7.3692	7.3078	7.3319	7.2959	7.3347	7.2596
0.5	6.7080	6.6401	6.6812	6.6301	6.5932	6.5653
0.6	6.1414	6.1058	6.0964	6.0861	6.0823	6.0609
0.8	7.3671	7.3592	7.3472	7.3042	7.2954	7.2759

Table 5: Infinity error norms $L_\infty \times 10^5$ for different values of α and β when θ = 0.5, N = 50 and Δt = 0.02.

β	$\alpha = 0$	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.8$
0	1.2496	1.2689	1.2947	1.3092	1.3258	1.3663
0.2	1.5983	1.6182	1.6424	1.6580	1.6740	1.7130
0.4	1.5442	1.5642	1.5903	1.6059	1.6230	1.6641
0.5	1.5136	1.5338	1.5603	1.5757	1.5934	1.6336
0.6	1.5050	1.5250	1.5508	1.5670	1.5835	1.6239
0.8	1.5250	1.5451	1.5708	1.5859	1.6036	1.6435

7. Conclusion

This work solves the space, time and space-time fractional KdV equations of order $0 < \alpha < 1$ using the cubic B-spline orthogonal collocation method in space and the finite difference method in time. The sparse nature of the cubic B-spline matrix and use of finite elements results is a computationally efficient scheme. The tabulated results show that the method achieves the best results which correspond to the least error and highest convergence rates when θ= 0.5. The method is unconditionally stable and convergent.

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